SEPTEMBER 1937.

XXXVI. *Some Applications of the Kirchhoff-Dirac Function in Problems involving Solutions of the Classical Wave Equation.* By T. LEWIS, *University College of Wales, Aberystwyth* [*].

1. *Introduction.*

IN his investigations on the propagation of light Kirchhoff [†] made use of a function $F(x)$ which vanishes in the region external to $\pm\delta$ and whose integral $\int_{-\infty}^{\infty} F(x)dx = 1$, δ being an arbitrarily small number. He treats this function as a continuous differentiable function, and finally arrives at a formula expressing the value of a function ϕ at a point in terms of its values on a closed surface and the values of its time derivative on this surface. He proved that such a function exists by actually giving its explicit form. He writes

$$F(x) = \frac{\mu}{\sqrt{\pi}} e^{\mu^2 x^2},$$

where μ can be made as large as we please.

Dirac [‡], in his theory representation of "states" and

" observables," finds it convenient to introduce a highly improper function, $\delta(x)$, into his analysis. This function is zero everywhere except at the point $x=0$, while at $x=0$ it is infinite to such a degree that $\int_{-\infty}^{\infty} \delta(x)\,dx=1$. He points out that the introduction of this improper function is not itself a source of lack of rigour in the theory, since all the results can be obtained, in a more cumbersome way, without its aid. But its introduction is certainly a great convenience. It is, however, doubtful whether all the properties attributed to $\delta(x)$ by Dirac are consistent with its definition. We refer to the property

$$x\delta(x)=0,$$

and the derivative one,

$$x\delta'(x)+\delta(x)=0.$$

These relations hold at all points except $x=0$, where there is a certain amount of ambiguity. This does not invalidate results obtained with the aid of $\delta(x)$, because, these, in general, involve the integral of $\delta(x)$. The attitude of mathematicians towards this function may be illustrated by a quotation from Weyl *, who calls it the " pathological " function $\delta(x)$. " Of course, there exists no such function, but it can be ' arbitrarily closely approximated ' by a function which vanishes everywhere except in a very small interval about $x=0$ and assumes very large values within this interval." This is precisely the definition of Kirchhoff's function. Weyl also points out that the passage to the limit is to take place *after*, not before, the other operations to be performed.

In the following paragraphs the Kirchhoff-Dirac function is used in order to prove a number of theorems. The main object is to illustrate the *utility* of the function in problems involving solutions of the classical wave equation. A number of the results are new, while others have been generalised. In §1 we state some of the properties of $\delta(x)$ and see to what extent they are valid when $\delta(x)$ is replaced by $K(x)$, which is the limiting form of an analytic function. In §3 we prove the Poisson-Liouville formula directly, and not as a corollary to Kirchhoff's formula. The fact that it can be deduced

* H. Weyl, ' The Theory of Groups and Quantum Mechanics,' p. 255.

from Kirchhoff's theorem is an accident which occurs in three dimensions only. In §4 we obtain a solution of the wave equation which reduces to prescribed functions at the instants t_1 and t_2. This result is new. In §4 we use a function which is the integral of the K function with respect to a parameter. This function is the precise analogue of Kirchhoff's function relative to the wave equation in two dimensions. With its aid we are able to obtain the result for two dimensions corresponding to the Poisson-Liouville formula for three dimensions. This result is identical with Hadamard's formula, and does not assume differentiability of the data. We also obtain another form by the " method of descent " from three dimensions.

In Part II. we show the convenience of the K function for the representation of the electromagnetic potentials and field intensities. In §5 we derive a form of the Lagrangian (and action) of two interacting electrons which reduces to Darwin's approximate form, and which is taken as the basis of the quantum theory of two interacting electrons. Of course, radiation terms are ignored. In §6 we obtain exact expressions for the radiation field of any continuous distribution of finite extension. Finally, in §7 we derive formulæ relative to the field of a rigidly electrified sphere moving without rotation, and verify the existence of the radiationless orbits found by Schott.

I.

General Theorems.

2. *Some Properties of the Kirchhoff-Dirac Function.*

According to Dirac the improper function $\delta(x)$ is zero for all values of x except $x=0$, where it becomes infinite in such a way that the integral $\int_{-\infty}^{\infty} \delta(x)\,dx$ is unity. Mathematicians are usually very uneasy about dealing with such a function in ordinary analysis, and it has been suggested that the δ function must be regarded as the limiting form of a nicely behaved continuous function. Regarded in this way, Dirac's improper function is equivalent to Kirchhoff's function, which is defined to be zero except when the argument lies between two arbitrarily small numbers $\pm\delta$. In what follows it will be more

convenient to regard the function, which we shall call the K function, as the limiting form of an analytic function, or, if preferred, as an analytic function containing a parameter h which we can make as small as we please without actually making it zero, so that the error in our formulæ is less than a prescribed quantity, however small that may be.

It has been suggested * that $\delta(x)$ may be regarded as the limiting form of

$$\frac{\sin 2\pi Nx}{\pi x} (N \to \infty) = \frac{1}{\pi} \times \text{Dirichlet's function.}$$

If we adopt this view, we have the theory of the Fourier integral to help us overcome any analytical difficulties.

For the Kirchhoff function, $K(x)$, the following have been suggested as suitable forms :

$$\underset{h \to 0}{\text{Lt}} \frac{h}{h^2 + x^2}, \quad \underset{h \to 0}{\text{Lt}} \frac{1}{\sqrt{\pi h}} e^{-x^2/h^2}.$$

In what follows, when we have need to refer to the explicit form of $K(x)$, we shall use the second form, which was suggested by Kirchhoff † himself. The reason for this is fairly obvious. Its range of validity is much wider than the former, and it is simpler in structure than Dirichlet's function.

Below we enumerate some of the formal properties attributed to Dirac's improper function $\delta(x)$, and see to what extent they hold for the Kirchhoff function,

$$K(x) = \underset{h \to 0}{\text{Lt}} \frac{1}{\sqrt{\pi h}} e^{-x^2/h^2}. \quad . \quad . \quad . \quad (2.1)$$

It will be convenient to state some of these properties in a slightly more general form than usual.

Firstly, we have

$$\left. \begin{array}{l} \delta(-x) = \delta(x), \qquad \delta(x) = 0 \text{ for } x \neq 0, \\[2mm] \int_{-a}^{b} \delta(x) \, dx = \begin{array}{l} 1 \text{ if } a \text{ and } b \text{ have the same sign} \\ 0 \text{ if } a \text{ and } b \text{ have different signs} \end{array} \end{array} \right\} \quad . \quad (2.2)$$

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* L. de Broglie, ' Théorie de la Quantification dans la nouvelle Mécanique,' p. 155.
† *Loc. cit.*

One easily verifies that for h sufficiently small

$$K(x) < \epsilon \quad \text{when } x \neq 0,$$

and

$$\int_{-a}^{b} K(x)\, dx < 1 + \epsilon \text{ or } \epsilon,$$

for a prescribed ϵ (according as whether a and b have the same or different signs).

Dirac also assumes

$$x\delta(x) = 0. \quad \ldots \ldots \quad (2.3)$$

This relation, however, cannot be rigorously true for $K(x)$. If one prescribes $x(\neq 0)$ one can always find h, so that

$$xK(x) < \epsilon.$$

But the maximum values of $|\, xK(x)\, |$ occur when $x = |\, h/\sqrt{2}\, |$. This maximum value is

$$|xK(x)|_m = 1/\sqrt{2\pi e}. \quad \ldots \ldots \quad (2.4)$$

This is independent of h, and is true however small h is. As $h \to 0$ this maximum is pushed towards the origin, so that, in the limit, $h = 0$

$$\left.\begin{array}{l} \underset{x \to 0}{\text{Lt}}\ xK(x) = 1/\sqrt{2\pi e}, \\[2mm] \underset{x \to 0}{\text{Lt}}\ -xK(-x) = -1/\sqrt{2\pi e}. \end{array}\right\} \quad \ldots \quad (2.5)$$

It follows that $xK(x)$ is discontinuous at the origin. But, instead of (2.3), we can write

$$\int_{-\infty}^{\infty} |xK(x)|\, dx = 0 \quad \ldots \ldots \quad (2.3)'$$

for

$$\int_{-\infty}^{0} xK(x)\, dx = -\underset{h \to 0}{\text{Lt}} \left[\frac{h}{2\sqrt{\pi}}\, e^{-x^2/h^2} \right]_{-\infty}^{0} = 0.$$

One easily verifies that

$$x^n K(x) = 0, \quad n > 1. \quad \ldots \ldots \quad (2.3)''$$

Next we have

$$\int_{-\infty}^{\infty} f(\xi)\, \delta(\xi - x)\, d\xi = \tfrac{1}{2}\{ f(x-0) + f(x+0) \}, \quad (2.6)$$

provided $f(x)$ is bounded and continuous in stretches and possesses a finite number of maxima and minima in any finite stretch.

If we write

$$f_h(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}h} e^{-(\xi-x)^2/h^2} f(\xi)\, d\xi,$$

one can always find h so small that

$$|f_h(x) - \tfrac{1}{2}\{f(x-0)+f(x+0)\}| < \epsilon.$$

The proof is elementary.

It is not necessary that $f(x)$ should be bounded. For example, $f(x)$ can approach infinity as any power (finite) of x without invalidating the last inequality. Poles will be considered at a later stage.

The differential coefficient of $\delta(x)$, which is still more improper, has, according to Dirac, the properties

$$\delta'(-x) = -\delta'(x), \quad x\delta'(x) + \delta(x) = 0. \qquad (2.7)$$

The first condition is satisfied by $K'(x)$. The left-hand side of the second equation is simply

$$\frac{d}{dx}(x\,\delta(x)).$$

Hence the second property cannot be true at the point $x=0$ when $\delta(x)$ is replaced by $K(x)$.

If $f(x)$ is bounded and continuous in stretches and if $f'(x)$ satisfies the same conditions,

$$\int_{-\infty}^{\infty} f(\xi)\,\delta'(\xi-x)\,d\xi = -\tfrac{1}{2}\{f'(x-0)+f'(x+0)\}$$
$$+ \{f(x-0)-f(x+0)\}\delta(0). \qquad (2.8)$$

The last term is absent if $f(x)$ is continuous at the point x.

Let

$$f_h'(x) = -\int_{-\infty}^{\infty} K'(\xi-x)f(\xi)\,d\xi,$$

where h is small, but not actually zero at this stage.

Then

$$-f_h'(x) = \Sigma\left[K(\xi-x)f(\xi)\right]_{a+}^{a-} - \int_{-\infty}^{\infty} K(\xi-x)f'(\xi)\,d\xi,$$

where $x=a$, b, etc. are points of discontinuity of $f(x)$.

If x itself happens to coincide with a or b etc.

$$-f_h'(x) = \{f(x-0)-f(x+0)\}K(0) - \int_{-\infty}^{\infty} K(\xi-x)f'(\xi)\,d\xi.$$

The integral on the right-hand side of this equation is similar to the one discussed above. In the limit $h \to 0$ we get a relation similar to (2.8).

Generally, if a function $f(x)$ is n times differentiable at the point x

$$f^n(x) = (-1)^n \int_{-\infty}^{\infty} f(\xi)\, \delta^n(\xi - x)\, d\xi. \quad . \quad . \quad (2.9)$$

This relation holds in the limit $h \to 0$, when $\delta(x)$ is replaced by $K(x)$.

Finally, suppose $f(x)$ has a pole of order n at $x = x_0$. For definiteness let

$$f(x) = \frac{g(x)}{(x - x_0)^n}, \quad . \quad . \quad . \quad . \quad (2.10)$$

where $g(x)$ is finite, continuous, etc. everywhere.

Let us write

$$f_h(x) = \frac{g(x)}{\{\pm \sqrt{(x - x_0)^2 + h^2 \gamma^2}\}^n}, \quad . \quad . \quad (2.10)'$$

where γ is a constant.

Then, if we remember that the sign of the square root changes on passing through the point x_0 ($-$ to the left and $+$ to the right).

$$\underset{h \to 0}{\mathrm{Lt}}\, f_h(x) = f(x). \quad . \quad . \quad . \quad . \quad (2.11)$$

Again, writing

$$\mathbf{f}_h(x) = \int_{-\infty}^{\infty} K(\xi - x) f_h(\xi)\, d\xi, \quad . \quad . \quad (2.12)$$

and proceeding to the limit $h = 0$, we easily prove that, unless $x = x_0$,

$$\underset{h \to 0}{\mathrm{Lt}}\, \mathbf{f}_h(x) = f(x). \quad . \quad . \quad . \quad . \quad (2.13)$$

It follows that we can represent a function $f(x)$ even when it has poles, provided we regard it as the limiting form of $f_h(x)$ defined by (2.10)' and (2.11).

Let us examine what happens when $x = x_0$.

$$\mathbf{f}_h(x_0) = \int_{-\infty}^{\infty} \frac{g(\xi)}{\{\pm \sqrt{(\xi - x_0)^2 + h^2 \gamma^2}\}^n} \cdot \frac{e^{-(\xi - x_0)^2 / h^2}}{\sqrt{\pi} h}\, d\xi.$$

Writing

$$\xi - x_0 = h\eta$$

$$\mathbf{f}_h(x_0) = \frac{g(x_0)}{h^n \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\eta^2}\, d\eta}{\{\pm \sqrt{\eta^2 + \gamma^2}\}^n} + O\left(\frac{1}{h^{n-1}}\right).$$

Now

$$\mathrm{I}(\gamma,\, n) = \int_{-\infty}^{0} \frac{e^{-\eta^2}\, d\eta}{\{\sqrt{\eta^2 + \gamma^2}\}^n} < \frac{1}{\gamma^n} \int_{-\infty}^{0} e^{-\eta^2}\, d\eta,$$

i. e., $\mathrm{I}(\gamma,\, n) < \sqrt{\pi}/2\gamma^n.$

Hence we can always choose γ such that

$$\mathrm{I}(\gamma,\, n) = \sqrt{\pi}/2.$$

With this understanding,

$$\mathbf{f}_h(x_0) = \frac{1}{2}\frac{g(x_0)}{h^n}\left[\{(-1)^n + 1\} + \mathrm{O}(h)\right]. \quad . \quad (2.14)$$

If n is even, $\underset{h=0}{\mathrm{Lt}}\, \mathbf{f}_h(x_0)$ has an infinity of the right order at $x = x_0$. If n is odd $\underset{h=0}{\mathrm{Lt}}\, \mathbf{f}_h(x_0)$ has an infinity of order $(n-1)$ at $x = x_0$. This is to be expected, for poles of odd order are of the nature of discontinuities, and the representation always gives the mean value at a point of discontinuity. The pole of lower order in this case is due to the *infinite* discontinuity.

It follows that functions having poles of any finite order can be replaced by functions, which, in the limit, behave very much in the same way, even at the poles ; these new functions can be represented by means of the K function. Thus a much wider range of functions can be represented than by means of the Fourier integral, which requires the absolute convergence of $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx.$

As an example, let

$$f(x) = x^n.$$

Then

$$f_h(x) = \frac{1}{\sqrt{\pi}\,h}\int_{-\infty}^{\infty} e^{-(\xi-x)^2/h^2}\,\xi^n\, d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\eta^2}(x+h\eta)^n\, d\eta,$$

where $h\eta = \xi - x$.

Performing the integration

$$f_h(x) = x^n + h\, \mathrm{P}_{n-1}(h,\, x),$$

where P_{n-1} is a polynomial in h, of degree $n-1$. Hence, proceeding to the limit,

$$\underset{h=0}{\mathrm{Lt}}\, f_h(x) = (x)^n.$$

Again,

$$f_h'(x) = \frac{1}{\sqrt{\pi h}} \int_{-\infty}^{\infty} \frac{2(\xi - x)}{h^2} e^{-(\xi-x)^2/h^2} \xi^n \, d\xi$$

$$= \frac{2}{\sqrt{\pi h}} \int_{-\infty}^{\infty} e^{-\eta^2} (x+h\eta)^n \eta \, d\eta = -\frac{e^{-\eta^2} (x+h\eta)^n}{\sqrt{\pi h}} \Big|_{-\infty}^{\infty}$$

$$+ \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} (x+h\eta)^{n-1} \, d\eta.$$

The integrated part contains a term

$$-\frac{e^{-\eta^2} x^n}{\sqrt{\pi h}} \Big|_{-\infty}^{\infty},$$

but it is identically zero however small h is. Or, suppose the limits of ξ are $\pm R$, where R is large. The limits of η are $\pm R/h$, approximately. It follows that the value of this term at a limit is $\dfrac{x^n e^{-R^2/h^2}}{\sqrt{h}}$, which vanishes with h. The other terms cause no difficulty. It follows that

$$\underset{h=0}{\mathrm{Lt}} f_h'(x) = n x^{n-1}.$$

In all our applications we shall operate upon $K(x)$ as if it were an analytic function and proceed to the limit $h=0$ at the final stage. We can regard the functions occurring in our formulæ as functions of h as well as the other variables ; but we are chiefly interested in the term independent of h, assuming, of course, that the functions can be expanded in powers of h.

3. *The Poisson-Liouville Theorem.*

This theorem is sometimes mistakenly regarded as a special case of Kirchhoff's theorem. But, in reality, it is complementary to the latter. We confine our attention in this paper to the Poisson-Liouville portion of the theorem, partly because the Kirchhoff part has been deduced with the aid of the K function by Kirchhoff himself, and his deduction is well known, and partly because we need the results obtained here for subsequent developments.

Let ϕ and ψ be two functions, finite, one valued and twice differentiable in a region Ω bounded internally

by a surface Σ and externally by a surface S, and let them both satisfy the wave equation in this region.

It follows that

$$\int_{t_1}^{t_2} \iiint_{\Omega} \left\{ \phi \left(\nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2} \right) - \psi \left(\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} \right) \right\} d\Omega \, d\tau = 0,$$

where (ξ, η, ζ, τ) are running coordinates, and the velocity of propagation is unity.

Partial integration gives

$$\int_{t_1}^{t_2} \iint_{\Sigma} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\Sigma \, d\tau = \int_{t_1}^{t_2} \iint_{S} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS \, d\tau$$

$$- \iiint \left(\phi \frac{\partial \psi}{\partial \tau} - \psi \frac{\partial \phi}{\partial \tau} \right) d\Omega \Big|_{t_1}^{t_2}, \quad . \quad (2.1)$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal.

The integral over S gives rise to the Kirchhoff formula. But here it will be assumed that when $(t_2 \geqslant t \geqslant t_1)$ the disturbance represented by ϕ is finite in extent, so that S can be chosen in such a way that both ϕ and $\frac{\partial \phi}{\partial n}$ are zero at all points of it. With this understanding we can omit the S integral in what follows.

We now introduce the elementary solution

$$\psi = \frac{1}{r} \left\{ \alpha K(r - t + \tau) + \beta K(r + t - \tau) \right\}, \quad . \quad (2.2)$$

where K is the function discussed in the previous paragraph,

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2,$$

(x, y, z) being a point enclosed by Σ and t any instant between t_1 and t_2. α, β are arbitrary constants.

The first term on the right-hand side represents a spherical pulse converging upon (x, y, z) at the instant t, having a strong maximum at $r = t - \tau$. The second term is a similar, diverging, pulse. To save unnecessary writing we suppress the second term in the actual derivation of our formula and write down the part of the formula to which it gives rise from analogy with the part due to the first term.

Let us first deal with the integral over Σ, namely,

$$I_\Sigma = \int_{t_1}^{t_2} \int\!\!\int_\Sigma \left\{ \phi \frac{\partial}{\partial n}\left(\frac{K}{r}\right) - \frac{K}{r}\frac{\partial \phi}{\partial n} \right\} d\Sigma\, d\tau. \quad (2.3)$$

Σ will be taken to be a sphere of arbitrarily small radius and centre (x, y, z), so that $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$, and hence,

$$I_\Sigma = \int_{t_1}^{t_2} \int\!\!\int_\Sigma \left\{ \phi\left(\frac{K'}{r} - \frac{K}{r^2}\right)\frac{K}{r} - \frac{\partial\phi}{\partial r} \right\} r^2\, d\omega\, d\tau,$$

$d\omega$ being an element of solid angle.

Partial integration of the first term with respect to τ gives

$$I_\Sigma = \int\!\!\int \phi K(r-t+\tau)r\, d\omega \Big|_{t_1}^{t_2}$$
$$- \int_{t_1}^{t_2} \int\!\!\int_\Sigma \left\{ \phi + r\frac{\partial\phi}{\partial\tau}\left(+\frac{\partial\phi}{\partial r}\right) \right\} K\, d\omega\, d\tau.$$

The integrated term is zero unless

$$t = t_1 + r \quad \text{or} \quad t_2 + r.$$

The second case is clearly ruled out, since $t_2 \geqslant t \geqslant t_1$. Again, we can make r arbitrarily small so that the difficulty arises only when $t \to t_1$. But then we have

$$rK(r) = 0,$$

however small r is, provided it is not identically zero.

It follows that the integrated term is always negligible.

Integrating the second term with respect to τ we get

$$I_\Sigma = -\int\!\!\int_\Sigma \left\{ \phi + r\left(\frac{\partial\phi}{\partial\tau} + \frac{\partial\phi}{\partial r}\right) \right\}_{\tau = t-r} d\omega.$$

Proceeding to the limit $r = 0$, we get, finally,

$$\underset{r=0}{\text{Lt}}\ I_\Sigma = -4\pi\phi(x, y, z, t). \quad \cdot \quad \cdot \quad \cdot \quad (2.4)$$

The volume integral is

$$I_\Omega = -\int\!\!\int\!\!\int \left\{ \phi K' - K\frac{\partial\phi}{d\tau} \right\} r\, dr\, d\omega\ \Big|_{t_1}^{t_2}.$$

Integrating the first term with respect to r gives

$$I_\Omega = -\int\int \phi r K(r_s - t + \tau)\, d\omega \Big|_{t_1}^{t_2}$$
$$+ \int\int\int \left\{ \frac{\partial}{\partial r}(r\phi) + r\frac{\partial\phi}{\partial\tau}\right\} K\, dr\, d\omega \Big|_{t_1}^{t_2}.$$

The integrated term is zero because ϕ is zero on S, or, what is equivalent, because on S, $r > t_2 - t_1$ so that K is zero for this value of r when t lies between t_1 and t_2.

Integration of the second term with respect to r gives

$$I_\Omega = -\int\int \left\{ \frac{\partial}{\partial r}(r\phi) + r\frac{\partial\phi}{\partial\tau}\right\} \Big|_{\substack{\tau = t_1 \\ r = t - t_1}} d\omega. \quad . \quad (2.5)$$

From (2.1), (2.4), and (2.6), it follows that

$$\phi(x, y, z, t) = \frac{1}{4\pi}\int\int \left\{ \frac{\partial}{\partial r}(r\phi) + r\frac{\partial\phi}{\partial\tau}\right\}_{\substack{\tau = t_1 \\ r = t - t_1}} d\omega, \quad . \quad (2.6)$$

which is the Poisson-Liouville formula.

It should be noticed that this formula presupposes the differentiability of ϕ at the instant $t = t_1$. But it can easily be generalized so as to accommodate discontinuities in $\frac{\partial\phi}{\partial r}$.

Now, if in the formula (2.2) for ψ we put $\alpha = 1$ and $\beta = 1$, we arrive at a corresponding formula for ϕ in terms of ϕ and $\frac{\partial\phi}{\partial\tau}$ at the subsequent instant t_2, namely,

$$\phi(x, y, z, t) = \frac{1}{4\pi}\int\int \left\{ \frac{\partial}{\partial r}(r\phi) - r\frac{\partial\phi}{\partial\tau}\right\} \Big|_{\substack{\tau = t_2 \\ r = t_2 - t}} d\omega. \quad . \quad (2.7)'$$

Or, more generally, we can express ϕ as a linear combination of the right-hand sides of (2.6) and (2.7), obtaining a formula which corresponds more closely to the analogous elliptic formula.

(2.7) and (2.7)' merely show that both ϕ and $\frac{\partial\phi}{\partial\tau}$ cannot be arbitrarily assigned at both instants t_1 and t_2. We shall now find a formula for ϕ when ϕ only is prescribed at t_1 and t_2.

It is required to find a function ϕ satisfying the wave equation and reducing to prescribed functions of the space coordinates at the instants t_1 and t_2.

This problem is the hyperbolic analogue of Dirichlet's problem for the elliptic equation. It can be solved with the aid of a function corresponding to the Green's function used in solving Dirichlet's problem. Such a function can be constructed by superposing elementary solutions involving the K function. In fact, a somewhat novel use is made of the method of images, bringing in to play both retarded and advanced potentials.

We see from equation (2.1) that our purpose is attained if we can find a function ψ which behaves like

$$\frac{K(r-t+\tau)}{r}$$

near the point (x, y, z), and vanishes when

$$\tau = t_1 \quad \text{and} \quad \tau = t_2.$$

One easily verifies that ψ is such a function if

$$\left.\begin{aligned}
2r\psi = &K(r-t+\tau) - K(r+t_{01}-\tau) \\
&+ K(r-t_{012}+\tau) - K(r+t_{0121}-\tau) + \ldots \\
&+ K(r+t-\tau) - K(r-t_{02}+\tau) \\
&+ K(r+t_{021}-\tau) - K(r-t_{0212}+\tau) + \ldots
\end{aligned}\right\} \quad . \quad (3.1)$$

where

$$\left.\begin{aligned}
t_{01}+t=2t_1, \quad t_{012}+t_{01}=2t_2, \quad t_{0121}+t_{012}=2t_1, \quad \text{etc.,} \\
t_{02}+t=2t_2, \quad t_{021}+t_{02}=2t_1, \quad t_{0212}+t_{021}=2t_2, \quad \text{etc.,}
\end{aligned}\right\} \quad . \quad (3.2)$$

Putting $\tau = t_1$ in (3.1) and using the relations (3.2), the terms of the first series cancel, the first with the second, the third with the fourth, etc. The first term of the second series is zero for $\tau = t_1$, because $r+t-t_1 > 0$ for all r when $t > t_1$, while the second term cancels the third, the fourth cancels the fifth, etc. Putting $\tau = t_2$, the first series behaves like the second when $\tau = t_1$, and the second like the first.

Again, when $r = 0$, and τ lies between t_1 and t_2, all terms on the right-hand side of (3.1) with the exception of the first in each series are zero.

It is easily seen that t_{01} is the image of t with respect to t_1, t_{012} the image of t_{01} with respect to t_2, etc. t_{02} is

the image of t with respect to t_2, t_{021}, the image of t_{02} with respect to t_1, etc. The suffix notation explains itself.

If, as before, the disturbance represented by ϕ is finite in extent in the interval (t_1, t_2), the double series (3.1) can be terminated at a finite stage.

Substituting for ψ in (2.1) and proceeding as before, we find the following series for ϕ.

$$4\pi\phi(x, y, z, t) = \tfrac{1}{2}\iint \{ \Phi_1(t-t_1) - \Phi_1(t_1-t_{01})$$
$$+ \Phi_1(t_{012}-t_1) - \ldots + \Phi_2(t_2-t)$$
$$- \Phi_2(t_{02}-t_2) + \Phi_2(t_2-t_{021}) - \ldots \} d\omega, \quad (3.3)$$

where

$$\Phi_1(r) = \frac{\partial}{\partial r}(r\phi)\Big|_{\tau=t_1}, \quad \text{and} \quad \Phi_2(r) = \frac{\partial}{\partial r}(r\phi)\Big|_{\tau=t_2}. \quad (3.4)$$

This completes the solution of our problem.

The present use of the method of images is not on a par with Volterra's * use of the method in connexion with solutions of the wave equation satisfying certain boundary conditions. Volterra's images are the usual space images of potential theory and he does not mention the above type of problem. He considers only those problems in which ϕ and $\dfrac{\partial \phi}{\partial t}$ are prescribed at $t=0$.

4. *The Solution of the Two-dimensional Problem.*

The two-dimensional problem corresponding to the Poisson-Liouville problem has engaged the attention of many mathematicians. The most usual form of the solution is that known as the Poisson-Parseval †. Hada-mard ‡, however, arrives at a slightly more general form of the solution which does not assume the differentiability of the function representing the initial distribution of ϕ. Owing to the singularity of the elementary solution adopted by Hadamard serious analytical difficulties arise if one attempts a straightforward procedure as in § 2. To overcome these difficulties Hadamard takes the characteristic

* V. Volterra, ' L'integration des équations différentialles,' p. 53.
† See Volterra, *loc. cit.* p. 51.
‡ Hadamard, ' Lectures on Cauchy's Problem, etc.' p. 208.

cone as part boundary of the region of integration. He then makes use of a somewhat paradoxical concept—" the finite part of an improper integral "—for his integrals become infinite when extended to the characteristic one, though, when properly grouped, the singularities cancel out altogether. Hadamard also points out that the solution of the problem involving $n-1$ dimensions can be derived from the solution of the n dimensional problem by the " method of descent," *i. e.*, by regarding the former as a special case of the latter and introducing some obvious transformations.

Let us, for example, assume that the ϕ and $\dfrac{\partial \phi}{\partial \tau}$ appearing on the right-hand side of (2.7) are independent of ζ, and write

$$z - \zeta = \sqrt{r^2 - \rho^2}.$$

In terms of ρ, θ, ζ (*i. e.*, cylindrical coordinates)

$$d\omega = \frac{\rho \, d\rho \, d\theta}{r\sqrt{r^2 - \rho^2}},$$

and

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial \rho} \frac{\partial \rho}{\partial r} + \frac{\partial \phi}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial r} = \frac{\rho}{r} \frac{\partial \phi}{\partial \rho}.$$

Hence

$$\frac{\partial}{\partial r}(r\phi) = \frac{\partial}{\partial \rho}(\rho\phi).$$

Substituting these expressions in (2.7), we get

$$\phi(x, y, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{t-t_1} \left\{ \frac{\frac{\partial}{\partial \rho}(\rho\phi)}{t - t_1} + \frac{\partial \phi}{\partial \tau} \right\} \frac{\rho \, d\rho \, d\theta}{\sqrt{(t - t_1)^2 - \rho^2}}.$$

$$\cdots \quad (4.1)$$

This formula is in some respects simpler than the Poisson-Parseval formula ; but it implies that ϕ is given as a differentiable function of the coordinates at time t_1.

Below we give another solution of the problem and arrive finally at Hadamard's own formula, which does not imply differentiability of the data. Our method, however, will be identical with that used in §2. We are enabled to do this because of our choice of elementary

solution, which has no singularities except at $r=0$.   On the characteristic cone it has a sharp maximum as in the three-dimensional case.   In fact, we derive our elementary solution from the previous one by the " method of descent." This does not mean that our procedure is equivalent to the above, where the " method of descent " is applied at the last stage.   Hadamard's elementary solution for the two-dimensional case can also be deduced from his elementary solution for the three-dimensional case by the " method of descent."   We can say that our two elementary solutions are related to one another in the same way as Hadamard's elementary solutions are related to one another.

Let ϕ and ψ be two functions satisfying the two-dimensional wave equation in the area A, bounded internally by a curve σ and externally by a curve s. We get a formula corresponding to (2.1), namely,

$$\int_{t_1}^{t_2}\int_{\sigma}\left\{\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right\}d\sigma\,d\tau$$
$$=-\int\int_{A}\left\{\phi\frac{\partial\psi}{\partial\tau}-\psi\frac{\partial\phi}{\partial\tau}\right\}dA\Big|_{t_1}^{t_2},\quad(4.2)$$

where the integral over s has been omitted for the same reason as in §2.

An essential difference between our method and Hadamard's is our arbitrary choice of region of integration. Hadamard's region of integration is partly bounded by the characteristic cone.

Our elementary integral is

$$\psi=\frac{1}{2}\int_{-\infty}^{\infty}\frac{K(\sqrt{r^2+\zeta^2}-t+\tau)}{\sqrt{r^2+\zeta^2}}d\zeta,\quad.\quad.\quad(4.3)$$

where

$$r^2=(x-\xi)^2+(y-\eta)^2,$$

(x,y) being a point within σ.

As we have already emphasized, the parameter h in K is finite at this stage, though as small as we please.   We proceed to the limit $h=0$ at the very end of our operations, provided we find that such a procedure is legitimate and does not lead to meaningless expressions.

It is convenient to introduce a new parameter

$$u=\sqrt{r^2+\zeta^2}-r,$$

so that

$$\psi = \int_0^\infty \frac{K(u+r-t+\tau)}{\sqrt{u(u+2r)}} \, du. \quad . \quad . \quad (4.3)'$$

Except in the neighbourhood of the characteristic cone $r = t - \tau$, ψ behaves like Hadamard's elementary solution

$$\psi_1 = \frac{1}{\sqrt{(t-\tau)^2 - r^2}}. \quad . \quad . \quad . \quad (4.4)$$

But on the characteristic cone itself ψ is finite whereas ψ_1 is infinite. The point $r = 0$, however, is a singularity of ψ and has to be excluded from the region of integration by the closed curve σ.

The essential difference between wave propagation in two and three dimensions is simply illustrated by our two elementary solutions. Clean propagation takes place in three dimensions, whereas in two dimensions the wave leaves a tail behind. We must expect the parameter u, or its equivalent, to remain in our final formulæ.

Hadamard's elementary solutions do not admit of *simple* physical interpretation, and, in particular, they do not put in evidence the difference between two- and three-dimensional propagation.

We require expressions for the derivatives of ψ in terms of the K function.

$$\begin{aligned}
\frac{\partial \psi}{\partial \tau} &= \int_0^\infty \frac{K'(u+r-t+\tau)}{\sqrt{u(u+2r)}} \, du \\
\frac{\partial \psi}{\partial r} &= \frac{\partial \psi}{\partial \tau} - \int_0^\infty \frac{u K(u+r-t+\tau)}{\{u(u+2r)\}^{3/2}} \, du
\end{aligned} \quad . \quad (4.5)$$

These functions are finite and continuous except at $r = 0$, and, except near $r = t - \tau$, they behave like the corresponding derivatives of ψ_1.

In order to get Hadamard's formula it is necessary to transform the integral in (4.5).

Consider the function

$$f_\epsilon = \int_\epsilon^\infty \frac{K' \, du}{\sqrt{u(u+2r)}}.$$

Partial integration gives

$$f_\epsilon = -\frac{K(\epsilon+r-t+\tau)}{\sqrt{\epsilon(\epsilon+2r)}} - \int_\epsilon^\infty K\frac{\partial}{\partial u}\Big(\frac{1}{\sqrt{u(u+2r)}}\Big)\,du.$$

But we can write

$$-\frac{1}{\sqrt{\epsilon(\epsilon+2r)}} = \int_\epsilon^\infty \frac{\partial}{\partial u}\Big(\frac{1}{\sqrt{u(u+2r)}}\Big)\,du.$$

Hence

$$f_\epsilon = \int_\epsilon^\infty \{K(u+r-t+\tau)-K(\epsilon+r-t+\tau)\}\,\frac{(u+r)du}{\{u(u+2r)\}^{3/2}}.$$

If we proceed to the limit $\epsilon=0$ we get two infinite integrals. But their difference, however, is finite.

Following Hadamard, we write

$$f = \operatorname*{Lt}_{\epsilon\to 0} f_\epsilon = \int_0^\infty \{K(u+r-t+\tau)-K(r-t+\tau)\}\frac{(u+r)du}{\{u(u+2r)\}^{3/2}}.$$

$$\text{. . . (4.6)}$$

Hadamard employs a special notation. He writes

$$f = \left|\overline{\int_0^\infty K(u+r-t+\tau)\,\frac{(u+r)du}{\{u(u+2r)\}^{3/2}}}\right.,$$

where the symbol $\left|\overline{}\right.$ implies the " finite part " of the integral to which it is applied.

It should be noted that our use of this concept, " finite part of an improper integral," is incidental, whereas the rôle played by it in Hadamard's method is fundamental on account of the actual singularity in his elementary solution.

From (4.5) and (4.6) we get

$$\frac{\partial\psi}{\partial\tau} = \int_0^\infty \{K(u+r-t+\tau)-K(r-t+\tau)\}\frac{(u+r)du}{\{u(u+2r)\}^{3/2}}$$

$$\frac{\partial\psi}{\partial r} = \int_0^\infty \{rK(u+r-t+\tau)-(u+r)K(r-t+\tau)\}$$

$$\times \frac{du}{\{u(u+2r)\}^{3/2}}.$$

$$\text{. . . (4.7)}$$

We now proceed to evaluate the two integrals in (4.2). If σ is a small circle, centre (x, y),

$$
\begin{aligned}
I_\sigma = \int_{t_1}^{t_2} \int_0^{2\pi} \int_0^\infty & \{(K(u+r-t+\tau) - K(r-t+\tau))r^2 \\
& - K(r-t+\tau)ur\} \frac{\phi \, du \, d\theta \, d\tau}{\{u(u+2r)\}^{3/2}} \\
& - \int_{t_1}^{t_2} \int_0^{2\pi} \int_0^\infty \frac{K(u+r-t+\tau)}{\sqrt{u(u+2r)}} \frac{\partial \phi}{\partial r} r \, du \, d\theta \, d\tau.
\end{aligned}
$$

We now integrate with respect to τ, so that we are entitled to make use of the properties of the K function as h approaches zero, ϕ, the only other function containing τ, behaves quite regularly everywhere.

The first term, which has to be taken together, becomes, on integration,

$$
\begin{aligned}
\int_0^{2\pi} \int_0^{t-t_1-r} & \{\phi(t-r-u) - \phi(t-r)\} \frac{r^2 \, du \, d\theta}{\{u(u+2r)\}^{3/2}} \\
& - \int_0^{2\pi} \int_{t-t_1-r}^\infty \frac{\phi(t-r)r^2 \, du \, d\theta}{\{u(u+2r)\}^{3/2}}.
\end{aligned}
$$

The first integral is the " finite part " of an improper integral. One verifies without difficulty that it vanishes with r. The same is true of the second integral.

The second term in I_σ becomes, on integration with respect to τ,

$$
- \int_0^{2\pi} \int_0^\infty \frac{\phi(t-r)r \, u \, du \, d\theta}{\{u(u+2r)\}^{3/2}} = - \int_0^{2\pi} \phi(t-r) \, d\theta.
$$

The third term vanishes with r. Hence, proceeding to the limit $r = 0$,

$$
I_\sigma = -2\pi\phi(x, y, t). \quad . \quad . \quad . \quad . \quad (4.8)
$$

Again,

$$
\begin{aligned}
I_A = \int_0^\infty \int_0^{2\pi} \int_0^\infty & \{K(u+r-t+t_1) \\
& - K(r-t+t)_1\} \frac{(u+r)r\phi \, du \, d\theta \, dr}{\{u(u+2r)\}^{3/2}} \\
& - \int_0^\infty \int_0^{2\pi} \int_0^\infty K(u+r-t+\tau)r \frac{\partial \phi}{\partial \tau} \frac{du \, d\theta \, dr}{\sqrt{u(u+2r)}}.
\end{aligned}
$$

Integrating the first term with respect to r, it becomes

$$\int_0^{2\pi} \int_0^{t-t_1} \left\{ \frac{(t-t_1+u)}{\{u(2t-2t_1-u)\}^{3/2}} \phi(t-t_1-u) \right.$$
$$\left. - \frac{(t-t_1+u)}{\{u(2t-2t_1+u)\}^{3/2}} \phi(t-t_1) \right\} (t-t_1) \, du \, d\theta$$
$$- \int_0^{2\pi} \int_{t-t_1}^{\infty} \frac{(t-t_1+u)(t-t_1)}{\{u(2t-2t_1+u)\}^{3/2}} \phi(t-t_1) \, du \, d\theta.$$

The first of these integrals is the sum of two infinite integrals, but their sum is finite.

If, in the first integral, we write

$$u(2t-2t_1-u) = v^2,$$

and in the second and third,

$$u(2t-2t_1+u) = v^2,$$

and adjust the limits of the last two, we find that the sum of the three may be written

$$\int_0^{2\pi} \int_0^{t-t_1} \{\phi(\rho)-\phi(t-t_1)\} \frac{(t-t_1) \, dv \, d\theta}{v^2}$$
$$- \int_0^{2\pi} \int_{t-t_1}^{\infty} \phi(t-t_1) \frac{t-t_1}{v^2} dv \, d\theta,$$

where, for brevity, we have written ρ for $\sqrt{(t-t_1)^2-v^2}$.

The value of the second of these integrals becomes, on integration with respect to v,

$$- \int_0^{2\pi} \phi(t-t_1) \, d\theta.$$

The last term in I_λ is

$$- \int_0^{2\pi} \int_0^{t-t_1} \frac{(t-t_1) \dfrac{\partial \phi}{\partial \tau}}{\sqrt{u(u+2r)}} \bigg|_{r=t-t_1-u} du \, d\theta$$
$$= - \int_0^{2\pi} \int_0^{t-t_1} \frac{(t-t_1) \dfrac{\partial \phi}{\partial \tau}}{\rho} \bigg|_{r=\rho} dv \, d\theta.$$

Hence

$$I_A = \int_0^{2\pi} \int_0^{t-t_1} \left\{ \frac{\phi(\rho) - \phi(t-t_1)}{v^2} - \frac{1}{\rho} \frac{\partial \phi}{\partial \tau}_{\tau=\rho} \right\} (t-t_1)\, dv\, d\theta \\ - \int_0^{2\pi} \phi(t-t_1)\, d\theta. \quad (4.9)$$

Equations (4.8) and (4.9) give

$$2\pi\phi(x, y, t) = -\int_0^{2\pi} \int_0^{t-t_1} \left\{ \frac{\phi(\rho) - \phi(t-t_1)}{v^2} - \frac{1}{\rho} \frac{\partial \phi}{\partial \tau}\bigg|_{\tau=\rho} \right\} \\ \times (t-t_1)\, dv\, d\theta + \int_0^{2\pi} \phi(t-t_1)\, d\theta. \quad (4.10)$$

This is identical with Hadamard's formula * if v is expressed in terms of ϕ. We have

$$v^2 = (t-t_1)^2 - \rho^2, \quad \text{and} \quad v\, dv = -\rho\, d\rho.$$

II.

The Representation of the Electromagnetic Potentials and Intensities.

5. *The Field of a Point Charge. The Lagrangian of Two Interacting Electrons.*

The Lienard-Wiechert point laws can be deduced in a very simple way by the aid of the K-function and without regarding a point charge as the limiting case of an extended charge as in the usual derivations.

The elementary solution

$$\frac{f(\tau)K(r-t+\tau)}{r}$$

represents the effect at (x, y, z, t) of an instantaneous point source at the point (ξ, η, ζ) at the instant τ, $f(\tau)$ being a measure of its strength.

The potential of a continuous linear, time series of point sources is

$$\phi = \int_{-\infty}^{\infty} \frac{f(\tau)K(r-t+\tau)\, d\tau}{r}. \quad . \quad . \quad . \quad (5.1)$$

---

* *Loc. cit.* p. 208, eqn. (60).

Remembering the properties of the K function, we get, on performing the integration with respect to τ,

$$\phi = \left[\frac{f(\tau)}{\kappa r}\right]_{\tau = t - r}, \quad \cdots \quad (5.2)$$

where κ is the usual Doppler factor.

The right-hand side of (5.2) includes the point laws of Lienard-Wiechert. There is a corresponding advanced potential solution.

It follows that the scalar potential of a moving point charge may be represented by

$$\phi = \int_{-\infty}^{\infty} \frac{K(r-t+\tau)}{r} d\tau, \quad \cdots \quad (5.3)$$

and the vector potential by

$$\mathbf{a} = \int_{-\infty}^{\infty} \frac{K(r-t+\tau)\,\mathbf{v}\,d\tau}{r} = \int_{-\infty}^{\infty} \frac{K(r-t+\tau)\,d\mathbf{r}}{r},$$

where $\mathbf{v}$ is the velocity and $d\mathbf{r}$ ($=\mathbf{v}\,d\tau$) an element of the path of the charge.

From these representations it follows at once that

$$\frac{\partial \phi}{\partial t} + \operatorname{div} \mathbf{a} = -\int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \tau}\left(\frac{K}{r}\right) d\tau + \left(\operatorname{grad}_\xi\left(\frac{K}{r}\right),\, d\mathbf{r}\right) \right\}$$
$$= -\int_{-\infty}^{\infty} d\left(\frac{K}{r}\right) = 0.$$

The electric and magnetic intensities are obtained with the aid of the formulæ

$$\mathbf{E} = -\operatorname{grad} \phi - \frac{\partial \mathbf{a}}{\partial t}, \quad \mathbf{H} = \operatorname{rot} \mathbf{a}. \quad \cdots \quad (5.5)$$

Hence

$$\mathbf{E} = -\int_{-\infty}^{\infty} \left\{ \left(\frac{K'}{r} - \frac{K}{r^2}\right)\mathbf{r}_0 - \frac{K'\mathbf{v}}{r} \right\} d\tau,$$

where $\mathbf{r}_0$ is a unit vector from the instantaneous position of the electron to the field point.

Partial integration of the first and third terms gives

$$\mathbf{E} = \int_{-\infty}^{\infty} \left\{ \frac{d}{d\tau}\left(\frac{\mathbf{r}_0 - \mathbf{v}}{r}\right) + \frac{\mathbf{r}_0}{r^2} \right\} K\, d\tau, \quad \cdots \quad (5.6)$$

while

$$\mathbf{H} = -\int_{-\infty}^{\infty} \left(\frac{\mathbf{K}'}{r} - \frac{\mathbf{K}}{r^2} \right) [\mathbf{r}_0, \ \mathbf{v}] \, d\tau$$

$$= -\int_{-\infty}^{\infty} \left\{ \frac{d}{d\tau} \cdot \frac{[\mathbf{r}_0, \ \mathbf{v}]}{r} + \frac{[\mathbf{r}_0, \ \mathbf{v}]}{r^2} \right\} \mathbf{K} \, d\tau. \quad (5.7)$$

The values of the integrals on the right-hand sides of (5.6) and (5.7) are equal to their respective integrands divided by κ and evaluated for $\tau = t - r$.

We will now obtain the well-known series for the retarded potential in terms of quantities evaluated at the instant t.

Provided the parameter h in K is finite we can write down the series

$$\mathbf{K}(r - t + \tau) = \sum_{0}^{\infty} \frac{\mathbf{K}^n(\tau - t)}{n!} r^n, \quad . \quad . \quad (5.8)$$

where $\mathbf{K}^n(\tau - t)$ is the nth derivative of $\mathbf{K}(\tau - t)$ with respect to τ and r is sufficiently small to ensure convergence.

Substituting this expression in (5.1) we get a series for ϕ, namely,

$$\phi = \sum_{0}^{\infty} \int_{-\infty}^{\infty} f(\tau) \mathbf{K}^n(\tau - t) r^{n-1} \, d\tau.$$

Bearing in mind the relation [in the limit $h \to 0$]

$$\int_{-\infty}^{\infty} \mathbf{F}(\tau) \mathbf{K}^n(\tau - t) \, d\tau = (-1)^n \mathbf{F}^n(t),$$

we get, finally,

$$\phi = \sum \frac{(-1)^n}{n!} \left[\left(\frac{d}{d\tau} \right)^n \{ f(\tau) r^{n-1} \} \right]_{\tau = t}. \quad . \quad . \quad (5.9)$$

We will next use the K function in order to transform the Lagrangian of two interacting electrons.

In a paper published by the author * it was shown that a field theory of electrons led to the following approximate form of the action

$$A = -\sum \int \{ m\sqrt{1 - v^2} + \tfrac{1}{2} e\phi - \tfrac{1}{2} e(\mathbf{v}, \ \mathbf{a}) \} \, dt \quad . \quad (5.10)$$

where $(\phi, \ \mathbf{a})$ are the potentials at the position of an electron due to the remaining electrons. Reaction of radiation,

* T. Lewis, " Electromagnetic Field Theory," Phil. Mag. (7) xx. p. 1000 (1935).

if such exists, is not included in (5.10). The appearance of the factor $\frac{1}{2}$ in the last two terms is rather unusual, but, as will be seen later, our action function is equivalent to Darwin's * approximation as far as that approximation goes. [For a full justification see the paper cited.]

Let there be two electrons e_1 and e_2 at r_1 and r_2 respectively, and let ϕ_{12} be the potential at e_2 due to e_1, and ϕ_{21} the potential at e_1 due to e_2. Then

$$\phi_{12}=\int_{-\infty}^{\infty}\frac{K(r_{12}-t_2+t_1)\,dt_1}{r_{12}}, \quad \phi_{21}=\int_{-\infty}^{\infty}\frac{K(r_{21}-t_1+t_2)\,dt_2}{r_{21}},$$
$$\qquad\qquad\qquad\qquad\qquad\qquad . \quad . \quad . \quad (5.11)$$

where
$$r_{12}=|\,r_2-r_1\,|,$$

with corresponding expressions for a_{12} and a_{21}.

The part of the action depending upon the potentials may therefore be written

$$A'=-\frac{e_1 e_2}{2}\int dt_1\int\frac{K(r_{21}-t_1+t_2)}{r_{21}}\{1-(v_1,\,v_2)\}\,dt_2$$

$$-\frac{e_2 e_1}{2}\int dt_2\int\frac{K(r_{12}-t_2+t_1)}{r_{12}}\{1-(v_2,\,v_1)\}\,dt_1$$

$$=-\frac{e_1 e_2}{2}\int\int\{K(r_{12}-t_1+t_2)+K(r_{12}+t_1-t_2)\}$$
$$\times\{1-(v_1,\,v_2)\}\frac{dt_1\,dt_2}{r}. \quad (5.12)$$

Integrating this expression with respect to t_2 and denoting by ϕ_{21}^a and ϕ_{21}^r the advanced and retarded potentials of e_2 at e_1, we get

$$A'=-\frac{e_1}{2}\int\{\phi_{21}^a+\phi_{21}^r-(v_1,\,a_{21}^a+a_{21}^r)\}\,dt_1. \quad (5.13)$$

It follows that the equations of motion of the electron e_1 may be derived from the modified action function

$$A_1=-\int\{m_1\sqrt{1-v_1^2}+\frac{e_1}{2}(\phi_{21}^a+\phi_{21}^r)-\frac{e_1}{2}(v_1,\,a_{21}^a+a_{21}^r)\}\,dt_1,$$
$$\qquad\qquad\qquad\qquad\qquad\qquad . \quad . \quad . \quad (5.14)$$

---

* C. G. Darwin, "Lagrangian Methods for High Speed Motion," Proc. Camb. Phil. Soc. xx. p. 56 (1920).

in which advanced and retarded potentials occur on an equal footing.

The same result would have been obtained if the advanced potentials had been introduced on an equal footing with the retarded potentials at the very beginning. In classical electromagnetism there has always been considerable reluctance about admitting advanced potentials, but the prejudice appears to be disappearing in the new mechanics. If we base the mechanics of electrons on the *Principle of Least Action* it is inevitable that advanced and retarded potentials should play symmetrical rôles, for the motion between two *universe* points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) depends upon a Lagrangian function, and conditions at these two points in an equal measure. It is not usual to find the path of least action subject to a prescribed value of the velocity at a given universe point.

We can obtain yet another form for the action. Using the expansion (5.8) we find that

$$K(r_{12} - t_1 + t_2) + K(r_{12} + t_1 - t_2) = 2 \sum_0 \frac{K^{2n}(t_2 - t_1) r_{12}^{2n}}{2n\,!}.$$

Substituting this series in (5.2) we get

$$A' = -e_1 e_2 \int \int \sum_0^\infty \frac{K^{2n}(t_2 - t_1) r_{12}^{2n-1}}{2n\,!} \{1 - (\mathbf{v}_1, \mathbf{v}_2)\} \, dt_1 \, dt_2. \quad (5.15)$$

Integration with respect to t_2 gives

$$A' = A_1' = -e_1 e_2 \int \sum_0^\infty \frac{1}{2n\,!} \left(\frac{\partial}{\partial t_2} \right)^{2n} [r_{12}^{2n-1}\{1 - (\mathbf{v}_1, \mathbf{v}_2)\}]_{t_2 = t_1} \, dt_1,$$
$$\qquad\qquad\qquad . \quad . \quad . \quad (5.16)$$

and integration with respect to t_1 gives

$$A' = A_2' = -e_1 e_2 \int \sum_0^\infty \frac{1}{2n\,!} \left(\frac{\partial}{\partial t_1} \right)^{2n} [r_{12}^{2n-1}\{1 - (\mathbf{v}_1, \mathbf{v}_2)\}]_{t_1 = t_2} \, dt_2.$$
$$\qquad\qquad\qquad . \quad . \quad . \quad (5.16)'$$

It must be borne in mind that $\dfrac{\partial}{\partial t_2}$ does not operate on (x_1, y_1, z_1, t_1) and $\mathbf{v}_1$, and that $\dfrac{\partial}{\partial t_1}$ does not operate on (x_2, y_2, z_2, t_2) and $\mathbf{v}_2$. After the differentiations have

been carried out the suffixes attached to the t coordinate can be suppressed.

If now we write ct for t and expand A_1' as far as terms in c^{-2} we get Darwin's approximation *.

To obtain a form of the action which involves the two electrons symmetrically we have merely to write

$$A' = \tfrac{1}{2}(A_1' + A_2')$$

so that

$$A = -\int \Big\{ m_1\sqrt{1-v^-} + m_2\sqrt{1-v_2^2}$$
$$+ \frac{e_1 e_2}{2} \Sigma \frac{1}{2n!} \Big(\frac{\partial^{2n}}{\partial t_2^{2n}} + \frac{\partial^{2n}}{\partial t_1^{2n}} \Big) [r_{12}^{2n-1}\{1-(\mathbf{v_1},\ \mathbf{v_2})\}] \Big|_{t_1,\,t_2=t} dt. \qquad . \quad . \quad . \quad (5.17)$$

If we expand the right-hand side of (5.17) as far as terms in c^{-2} and add a term

$$\frac{df}{dt} = \frac{e_1 e_2}{4} \frac{d}{dt} \Big\{ \frac{(\mathbf{r_2}-\mathbf{r_1},\ \mathbf{v_2}-\mathbf{v_1})}{r} \Big\}, \quad . \quad . \quad (5.18)$$

which contributes nothing to the equations of motion, we get Darwin's approximation in symmetrical form †, which, incidentally, is the basis of Breit's ‡ approximate theory for two interacting electrons in quantum mechanics. Our exact formula (5.17) would probably not be in agreement with higher approximations obtained by Darwin's method. For, although he does not give a clear indication how to proceed to higher approximations, it is not likely that his Lagrangian would contain only even powers of c^{-2} as in the present case. The absence of a term in c^{-1} is explained by the fact that

$$\frac{\partial^n}{\partial t^n}(r^{n-1}) = 0, \qquad \text{when } n=1.$$

One rather remarkable, and also convenient, feature of Darwin's approximation is the fact that it contains only coordinates and velocities. Accelerations are removed by adding a term of the type (5,18). This simplification does not persist in higher approximations.

* *Loc. cit.* (3.3).
† *Loc. cit.* (3.4).
‡ Breit, Phys. Review, xxxiv. p. 553 (1929) ; xxxvi. p. 383 (1930) ; xxxix. p. 616 (1932).

6. *The Field of Continuous Space Distributions.*
Exact Formulæ for the Radiation.

In this case it is convenient to adopt the Eulerian method rather than the Lagrangian method. If $\rho(\xi, \eta, \zeta, \tau)$ be the density of electric charge and $\mathbf{C}(\xi, \eta, \zeta, \tau)$ the density of electric current, the scalar and vector potentials at (x, y, z, t) will be

$$\left.\begin{aligned}
\phi &= \int_{-\infty}^{\infty} \int_{\Omega} \frac{\rho \mathrm{K}(r-t+\tau)\,d\Omega\,d\tau}{r}, \\
\mathbf{a} &= \int_{-\infty}^{\infty} \int_{\Omega} \frac{\mathbf{C}\mathrm{K}(r-t+\tau)\,d\Omega\,d\tau}{r}.
\end{aligned}\right\} \quad . \quad . \quad (6.1)$$

If (x, y, z, t) is a point occupied by electric charge, the non-homogeneous equations are satisfied at that point.

Integration with respect to τ yields the well-known integrals of Lorentz, namely,

$$\phi = \int \frac{\rho_{t-r}}{r}\,d\Omega, \qquad \mathbf{a} = \int \frac{\mathbf{C}_{t-r}}{r}\,d\Omega. \quad . \quad . \quad (6.2)$$

The electric and magnetic forces are given by

$$\left.\begin{aligned}
\mathbf{E} &= \int_{-\infty}^{\infty} \int_{\Omega} \left\{ \frac{1}{r} \frac{\partial}{\partial \tau}(\rho \mathbf{r}_0 - \mathbf{C}) + \frac{\rho \mathbf{r}_0}{r^2} \right\} \mathrm{K}(r-t+\tau)\,d\Omega\,d\tau, \\
\mathbf{H} &= -\int_{-\infty}^{\infty} \int_{\Omega} \left\{ \frac{1}{r} \frac{\partial}{\partial \tau}[\mathbf{r}_0, \mathbf{C}] + \frac{1}{r^2}[\mathbf{r}_0, \mathbf{C}] \right\} \\
&\qquad\qquad\qquad\qquad \times \mathrm{K}(r-t+\tau)\,d\Omega\,d\tau,
\end{aligned}\right\}$$

$$. \quad . \quad . \quad (6.3)$$

where $\mathbf{r}_0$ is a unit vector having the direction of $\mathbf{r}$, and discontinuities in ρ and $\mathbf{C}$, regarded as functions of τ, have been ignored. Such discontinuities, when they exist, give rise to intense electric and magnetic impulses perpendicular to the direction of propagation. There is also a longitudinal component of the electric impulse, but its total effect at a distant point is nil.

We will apply the above formulæ to calculate the radiation field of a distribution such as a Schrödinger distribution. For this purpose we assume that the charge and current outside a sphere of radius b is negligibly small, where b is *not* small in comparison with the wave-length of the emitted radiation, as is usually assumed when

calculating the radiation field. We neglect terms in
r^{-2} from the beginning. In K we can write $r-\xi$ for r,
where r now denotes the distance of the field point (x, o, o)
from the origin of coordinates. (We can always choose
our axes so that the field point lies on the x-axis.)

Again, for a field of the type to be considered,

$$\rho = -\operatorname{div} \mathbf{P}, \quad \mathbf{C} = \frac{\partial \mathbf{P}}{\partial \tau}, \quad . \quad . \quad . \quad (6.4)$$

i. e., the total charge is zero, so that the medium (or space)
may be regarded as polarized.

With these assumptions, the distant field is given by

$$\mathbf{E} = -\frac{1}{r} \int_{-\infty}^{\infty} \int_{\Omega} \left\{ \frac{\partial}{\partial \tau} \left(\mathbf{r}_0 \operatorname{div} \mathbf{P} + \frac{\partial \mathbf{P}}{\partial \tau} \right) \right\} \\ \times \mathrm{K}(r-\xi-t+\tau) \, d\Omega \, d\tau,$$

where the volume integral can again be extended through-
out the whole of space without appreciable error.

The first term may be written

$$-\frac{\mathbf{r}_0}{r} \int_{-\infty}^{\infty} \int_{\Omega} \left\{ \frac{\partial \dot{\mathrm{P}}_\xi}{\partial \xi} + \frac{\partial \dot{\mathrm{P}}_\eta}{\partial \eta} + \frac{\partial \dot{\mathrm{P}}_\zeta}{\partial \zeta} \right\} \\ \times \mathrm{K}(r-\xi-t+\tau) \, d\xi \, d\eta \, d\zeta \, d\tau,$$

where the dot denotes differentiation with respect to τ.

Integrating by parts and ignoring terms which vanish
at infinity, this expression becomes

$$-\frac{\mathbf{r}_0}{r} \int_{-\infty}^{\infty} \int_{\Omega} \dot{\mathrm{P}}^\xi \mathrm{K}'(r-\xi-t+\tau) \, d\Omega \, d\tau \\ = \frac{\mathbf{r}_0}{r} \int_{-\infty}^{\infty} \int_{\Omega} \ddot{\mathrm{P}}_\xi \mathrm{K} \, d\Omega \, d\tau.$$

Hence

$$\mathbf{E} = \frac{1}{r} \int_{-\infty}^{\infty} \int_{\Omega} \{\mathbf{r}_0 \ddot{\mathrm{P}}_\xi - \ddot{\mathbf{P}}\} \mathrm{K}(r-\xi-t+\tau) \, d\Omega \, d\tau$$

$$= -\frac{1}{r} \int_{-\infty}^{\infty} \int_{\Omega} \ddot{\mathbf{P}}_\perp \mathrm{K}(r-\xi-t+\tau) \, d\Omega \, d\tau, \quad . \quad (6.5)$$

where $\ddot{\mathbf{P}}_\perp$ is the component of $\ddot{\mathbf{P}}$ which is at right angles
to $\mathbf{r}_0$.

As usual, one finds that

$$\mathbf{H} = [\mathbf{r}_0, \mathbf{E}], \quad . \quad . \quad . \quad . \quad . \quad (6.6)$$

i. e., **E**, **H** are perpendicular to one another and to $\mathbf{r}_0$. In addition, the magnitude of **H** is equal to that of **E**.

Integrating the right-hand side of (6.5) and writing

$$\mathbf{I} = -\int_\Omega \ddot{\mathbf{P}}_\perp(\xi, \eta, \zeta, t-r+\xi)\, d\Omega, \quad . \quad . \quad (6.7)$$

$$\mathbf{E} = \mathbf{I}/r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.8)$$

It follows that the Poynting flux is

$$\Pi = \frac{1}{4\pi}\,\mathbf{I}^2.$$

It should be noted that it is not necessary to assume a simple periodic distribution in order to arrive at these results. If we are concerned with dipole radiation only

$$\mathbf{I} \backsimeq \mathbf{I}_0 = \int_\Omega \mathbf{P}_\perp(\xi, \eta, \zeta, t-r)\, d\Omega. \quad . \quad . \quad (6.9)$$

If a better approximating is required,

$$\mathbf{I} = \int_\Omega \{\ddot{\mathbf{P}}_\perp + \xi \dddot{\mathbf{P}}_\perp + \tfrac{1}{2}\xi^2 \ddddot{\mathbf{P}}_\perp + \ldots\}\Big|_{\tau = t-r}\, d\Omega. \quad . \quad (6.10)$$

If **P** is an odd function of (ξ, η, ζ) there is no dipole radiation. In this case all terms involving even powers of ξ in (6.10) vanish and

$$\mathbf{I} = \int_\Omega \xi \dddot{\mathbf{P}}_\perp\Big|_{\tau = t-r}\, d\Omega, \quad \text{approximately.} \quad . \quad (6.11)$$

If, on the other hand, **P** is an even function of the coordinates, the terms involving odd powers of ξ in (6.10) vanish on integration.

For a simple Schrödinger field

$$\ddot{\mathbf{P}} = -\frac{4\pi^2}{\lambda^2}\,\mathbf{P}. \quad . \quad . \quad . \quad . \quad (6.12)$$

It follows that in this case

$$\mathbf{I} = -\frac{4\pi^2}{\lambda^2}\int_\Omega \mathbf{P}_\perp \cos\frac{2\pi(t-r+\xi)}{\lambda}\, d\Omega, \quad . \quad (6.10)'$$

where $\mathbf{P}_\perp$ is now independent of t.

This expression shows that corrections for the dipole field will proceed in powers of b/λ, b being the radius of the sphere containing the oscillatory charge.

If we write

$$\mathbf{p}' = \int_\Omega \mathbf{P} \cos\frac{2\pi\xi}{\lambda}\, d\Omega, \qquad \mathbf{p}'' = \int \mathbf{P} \sin\frac{2\pi\xi}{\lambda}\, d\Omega, \quad (6.13)$$

$$\mathbf{I} = -\frac{4\pi^2}{\lambda^2}\left\{ \mathbf{p}'_\perp \cos\frac{2\pi(t-r)}{\lambda} - \mathbf{p}''_\perp \sin\frac{2\pi(t-r)}{\lambda} \right\}. \quad (6.14)$$

It follows that when $\mathbf{P}$ is an even function of the coordinates

$$\mathbf{I} = \mathbf{I}' = -\frac{4\pi^2}{\lambda^2} \mathbf{p}'_\perp \cos\frac{2\pi(t-r)}{\lambda}, \quad . \quad . \quad (6.15)$$

and when $\mathbf{P}$ is an odd function of coordinates and dipole radiation is absent

$$\mathbf{I} = \mathbf{I}'' = \frac{4\pi^2}{\lambda^2} \mathbf{p}''_\perp \sin\frac{2\pi(t-r)}{\lambda} . \quad . \quad . \quad (6.16)$$

From (6.14) we conclude that when dipole radiation exists, the effect of the $\mathbf{p}''$ term is to increase the intensity of radiation and also to change its phase.

Writing

$$\mathbf{p}''_\perp = \mathbf{p}'_\perp \tan\epsilon,$$

$$\mathbf{I} = -\frac{4\pi^2}{\lambda^2} \mathbf{p}'_\perp \sec\epsilon \cos\frac{2\pi(t-r+\epsilon)}{\lambda}; \quad . \quad (6.14)'$$

ϵ, of course, depends upon the direction $\mathbf{r}_0$.

Again, from (6.13) we see that $\mathbf{p}'$ is an even function of λ^{-1} while $\mathbf{p}''$ is an odd function. It follows that the amplitude of $\mathbf{I}$ in (6.14) is always an even function of λ^{-1}.

These results were obtained by the author in 1933. They are equivalent to Hulme's * results. Hulme, however, assumes a simple periodic field from the commencement, while our results (6.4) and (6.10) are independent of that assumption.

7. *The Potentials of Surface Distributions.*

The general expressions for potentials of a charged surface moving parallel to itself are

$$\left.\begin{array}{l} \phi = \displaystyle\int_{-\infty}^{\infty} \int_{S} \frac{\sigma(u,\, v,\, \tau) \mathrm{K}(r-t+\tau)}{r} \, \mathrm{H}(u,\, v) \; du \; dv \; d\tau, \\[3mm] \mathbf{a} = \displaystyle\int_{-\infty}^{\infty} \int_{S} \frac{\sigma \mathbf{V} \mathrm{K}(r-t+\tau)}{r} \, \mathrm{H}(u,\, v) \; du \; dv \; d\tau, \end{array}\right\} \quad (7.1)$$

where $(u,\ v)$ are parameters on the surface so that $d\mathrm{S} = \mathrm{H}\,du\,dv$ and $(\xi,\,\eta,\,\zeta)$ are functions of $(u,\,v,\,\tau).\,\sigma$, the surface density, will, in general, depend upon τ. $\mathbf{V}$ can be resolved into two components,

$$\mathbf{V} = \mathbf{V}'(\tau) + \mathbf{V}''(u,\,v,\,\tau),$$

where $\mathbf{V}'$ is the velocity of the surface and $\mathbf{V}''$ the velocity of the charge relative to the surface.

The electric and magnetic expressions are obtained as before. As an illustration we shall consider the field of a rigidly charged sphere which has been treated in detail by G. A. Schott *, who uses his own Fourier Integral representation. In this case σ is constant and $\mathbf{V}''$ is zero. We thus obtain

$$\phi = \sigma a^{2} \int_{-\infty}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{\mathrm{K}(r-t+\tau)}{r} \sin u \; dv \; d\tau,$$

with a corresponding expression for $\mathbf{a}$. (a is used for the radius of the sphere.)

By proper choice of axis

$$r^{2} = \mathrm{R}^{2} + a^{2} - 2\mathrm{R}a \cos u, \qquad r\,dr = \mathrm{R}a \sin u \; du,$$

where R is the distance from the centre of the sphere.

Making this change and integrating with respect to v

$$\phi = 2\pi\sigma a \int_{-\infty}^{\infty} \frac{d\tau}{\mathrm{R}} \int_{|\mathrm{R}-a|}^{\mathrm{R}+a} \mathrm{K}(r-t+\tau) \, dr, \quad . \quad . \quad (7.2)$$

with a corresponding expression for $\mathbf{a}$. r is now to be regarded as a parameter. In the case of external points we can change the parameter so that its limits are independent of τ.

Writing

$$r - \mathrm{R} = u,$$

$$\phi = \frac{e}{2a} \int_{-\infty}^{\infty} \frac{d\tau}{\mathrm{R}} \int_{-a}^{a} \mathrm{K}(u + \mathrm{R} - t + \tau) \, du, \quad . \quad . \quad (7.3)$$

* G. A. Schott, " The Electromagnetic Field of a Moving, Uniformly and Rigidly Electrified Sphere and its Radiationless Orbits," Phil. Mag. (7) xv. p. 752 (1933). Also Proc. Roy. Soc. A, clvi. p. 471 (1936).

and $\qquad \mathbf{a} = \dfrac{e}{2a} \displaystyle\int_{-\infty}^{\infty} \dfrac{\mathbf{V} \, d\tau}{\mathrm{R}} \int_{-a}^{a} \mathrm{K}(u + \mathrm{R} - t + \tau) \, du, \quad \cdot \quad (7.4)$

e being the total charge.

We get different forms for the potentials according as to whether we integrate first with respect to u and then with respect to τ or *vice versa*. Integrating with respect to u we get

$$\phi = \frac{e}{2a} \int_{t-\mathrm{R}-a}^{t-\mathrm{R}+a} \frac{d\tau}{\mathrm{R}} = \frac{e}{2a} \int_{\mathrm{Q}_1}^{\mathrm{Q}_2} \frac{ds}{\mathrm{VR}},$$

$$\mathbf{a} = \frac{e}{2a} \int_{t-\mathrm{R}-a}^{t-\mathrm{R}+a} \frac{\mathbf{V} \, d\tau}{\mathrm{R}} = \frac{e}{2a} \int_{\mathrm{Q}_1}^{\mathrm{Q}_2} \frac{d\mathbf{s}}{\mathrm{R}},$$

which are Schott's formulæ, Q_1, and Q_2 being points on the path of the centre of the sphere.

Integration with respect to τ gives

$$\phi = \frac{e}{2a} \int_{-a}^{a} \frac{du}{[\kappa \mathrm{R}]_{\tau = t - \mathrm{R} - u}}, \qquad \mathbf{a} = \frac{e}{2a} \int_{-a}^{a} \left[\frac{\mathbf{V}}{\kappa \mathrm{R}} \right]_{\tau = t - \mathrm{R} - u} du,$$

$$\cdot \quad \cdot \quad \cdot \quad (7.6)$$

where κ is the Doppler factor referring to the centre of the sphere. From the formulæ (7.5) Schott is able to deduce the possibility of radiationless orbits—it is only necessary that Q_1 and Q_2 should coincide.

We shall verify that this is so by actually calculating the radiation field.

Neglecting terms in r^{-2} we find from (7.3) and (7.4) that

$$\mathbf{E} = -\int_{-\infty}^{\infty} \frac{\mathbf{R}_0 - \mathbf{V}}{\mathrm{R}} \, d\tau \int_{-a}^{a} \mathrm{K}'(u + \mathrm{R} - t + \tau) \, du$$

$$= -\int_{-\infty}^{\infty} \frac{(\mathbf{R}_0 - \mathbf{V})}{\mathrm{R}} \{ \mathrm{K}(\mathrm{R} + a - t + \tau) - \mathrm{K}(\mathrm{R} - a - t + \tau) \} \, dt$$

$$= \left[\frac{\mathbf{R}_0 - \mathbf{V}}{\kappa \mathrm{R}} \right]_{\tau = t - \mathrm{R} - a}^{\tau = t - \mathrm{R} + a}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7.3)$$

Hence, if the motion of the centre of the sphere is periodic, with period $2a$ ($2a/c$ in the usual units) or any integral multiple thereof, $\mathbf{E}$ is zero. The same will be true of $\mathbf{H}$. In other words, there is no radiation when the sphere has such a motion.

XXXVII. *Experiments on Supraconductors in Alternating Magnetic Fields. By* J. G. DAUNT, *The Clarendon Laboratory, Oxford* *.

THE disturbance of supraconductivity by a magnetic field has been studied in detail by various magnetic methods [1] during the last few years. In all these methods the magnetic induction of the specimen was investigated, and the question has arisen how a specimen will behave which is partly in the supraconductive state and partly in the normal state. It has been shown by London [2] and Peierls [3] independently that in a body of simple geometrical shape an " intermediate " state has to be expected as soon as the threshold field is exceeded at any part of the surface of the supraconductor. Though the actual constitution of the " intermediate " state is still somewhat uncertain, it can be said that in it there will be no co-existence of macroscopic supraconductive and normal regions. Mendelssohn and Pontius [4] have shown that while this is actually true for a sphere, the magnetic phenomena observed in a body of more complicated shape (*e. g.*, a short cylinder) indicate that macroscopic supraconductive and normal (or intermediate) regions are in equilibrium in the field (or temperature' range in which the specimen passes from the supra conductive to the normal state.

It is the object of this paper to investigate the transition under conditions where we can expect co-existence of normal and supraconductive material, and to study the formation and shape of these regions in different phases of the destruction and reappearance of supraconductivity.

An additional purpose of the experiments was to investigate in detail the time-effects which, as was shown by Mendelssohn and Pontius [4], occur after changes of temperature or magnetic field if macroscopic supraconductive and normal regions are in co-existence.

In the method employed in these experiments the shielding effect of the persistent currents in a supraconductor was used as an indication of the existence of supraconductive regions. It follows electro-dynamically that changes of magnetic or electric field cannot be trans-

mitted through a closed surface of infinite conductivity, but will be compensated exactly by persistent currents in the surface. In order to approach ideal conditions as far as possible electromagnetic changes inside a long hollow cylinder of supraconductive material placed both in longitudinal and transverse magnetic fields were observed. This form of specimen was advisable as the factor of demagnetisation is simple (zero in longitudinal fields and 2π in transverse fields), and it allowed us to reduce the problem to a two-dimensional one by considering the cross-section of the cylinder as a closed supraconductive circle. We were also certain to get co-existence of supraconductive and normal material when destroying supraconductivity with transverse fields.

Method.

The procedure adopted in these experiments was to place the long hollow cylinder in an alternating magnetic field of the order of 1 gauss. When the specimen was in the supraconductive state it completely shielded the inside from any changes of field. When supraconductivity was destroyed by applying an external D.C. magnetic field at constant temperature, then the exterior alternating magnetic field could penetrate the wall of the cylinder and an alternating field (of the same frequency, but different in phase) would be set up in the core. This was measured by placing within the core a coil of many turns of fine wire with axis parallel to the external A.C. field, and measuring the alternating e.m.f. induced in this coil with a suitable amplifier and rectifier.

Experimental Arrangements.

The specimens were made of very pure poly-crystalline tin (Hilger " H.S." Brand, Lab. No. 10,000) of purity 99·996 per cent. cast in German silver moulds in air. The German silver was found to have no effect on the surface, and after casting it was peeled off. The cylinders were 8 cm. long and 1 cm. outside diameter with a wall thickness of 1 mm.

A homogeneous alternating magnetic field was produced by a small Helmholtz coil situated outside the apparatus at room-temperature, which will be referred to later as the primary. We employed a frequency of 50 cycles per second, which was chosen on account of the small shielding

effect of the hollow cylinder, when in the normal state, to fields of this frequency. For when the cylinder is in the normal state, the alternating field in the interior is smaller in intensity than the exterior field due to losses by currents induced in the surface of the cylinder. The magnitude of this shielding effect is dependent on the frequency of the exterior field as well as on the geometry of the system (thickness of the cylinder, etc.) and on the specific conductivity of the material from which the cylinder is made. We found that by making the frequency as small as 50 cycles per second the intensity of the alternating field in the interior was not diminished excessively by these losses.

It was also essential that the alternating field should cause no breakdown of supraconductivity due to its frequency alone [5]. Consequently a preliminary experiment was carried out by measuring at constant temperature the induction in a tin sphere [6] when placed in a homogeneous D.C. magnetic field with an A.C. magnetic field (amplitude of 3·8 gauss) of frequency 50 cycles superposed. It was found that the threshold value and induction in the supraconductive state was independent of the presence of the A.C. field, showing that the frequency of 50 cycles was one which caused no breakdown of supraconductivity.

Different amplitudes of A.C. field were employed within the limits 0·42 gauss to 3·8 gauss.

Into the hollow core of the cylinder was placed a coil, which will be referred to later as the secondary. Two types of secondary were employed : one of 3000 turns of copper wire wound on a glass tube 0·5 cm. diameter with axis parallel to the axis of the cylinder, the other of 400 turns wound on a rectangular copper frame with axis perpendicular to the axis of the cylinder.

Care was taken that both these secondary coils were situated well in the middle of the long cylinders to avoid any spurious effects which might occur near the ends.

The specimen with its secondary coil was completely immersed in liquid helium produced in a liquefier * working on the expansion method [7].

The secondary was connected to an amplifier and rectifier.

* A detailed description of this helium liquefier will be given in the Proc. Phys. Soc.

Two types of amplifier were used, the second of which was found to have superior sensitivity and stability.

The first consisted of a two-valve resistance coupled amplifier leading to an anode bend type rectifier, the output from which was connected to a millivoltmeter balanced to read zero for zero input.

The second amplifier used only one transformer coupled amplifying valve which was followed by a rectifier arranged as part of a Wheatstone's Bridge having a galvanometer as measuring instrument. A diagram of this circuit is given in fig. 1. The Bridge arrangement was not used as a null method, but the actual deflexion of the galvanometer was taken as a measure of the alternating e.m.f. input.

Fig. 1.
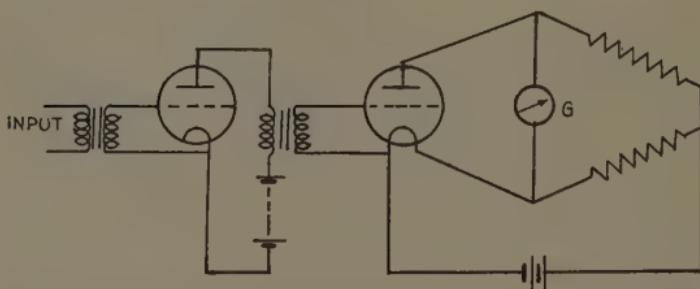
The characteristic curves of both amplifiers are shown in fig. 2, from which all necessary corrections to our results have been made.

The characteristic curve of amplifier I was made with the anode resistance of the last valve, across which the millivoltmeter was placed, equal to 140 ohms. By varying this, a convenient control was obtained on the sensitivity. If it, however, was greater than 500 ohms, the stability of the amplifier was not sufficiently good.

The characteristic curve of amplifier 2 was made with the galvanometer shunted to give only 1/20th of its maximum sensitivity. In practice the galvanometer was always shunted (usually to give 2/5ths of its maximum sensitivity), so that voltages of 1.4×10^{-6} (R. M. S.) gave 1 cm. deflexion on the galvanometer. The voltage sensitivity of the galvanometer, without shunts, was 2.5 mm. per microvolt.

The good approximation of the characteristic curve of this amplifier to linearity makes it very convenient in use.

The D.C. fields were produced both for the longitudinal and transverse measurements by coils which gave a field homogeneous to 1 per cent. over the whole volume of the specimen. The current was taken from accumulators.

Fig. 2.
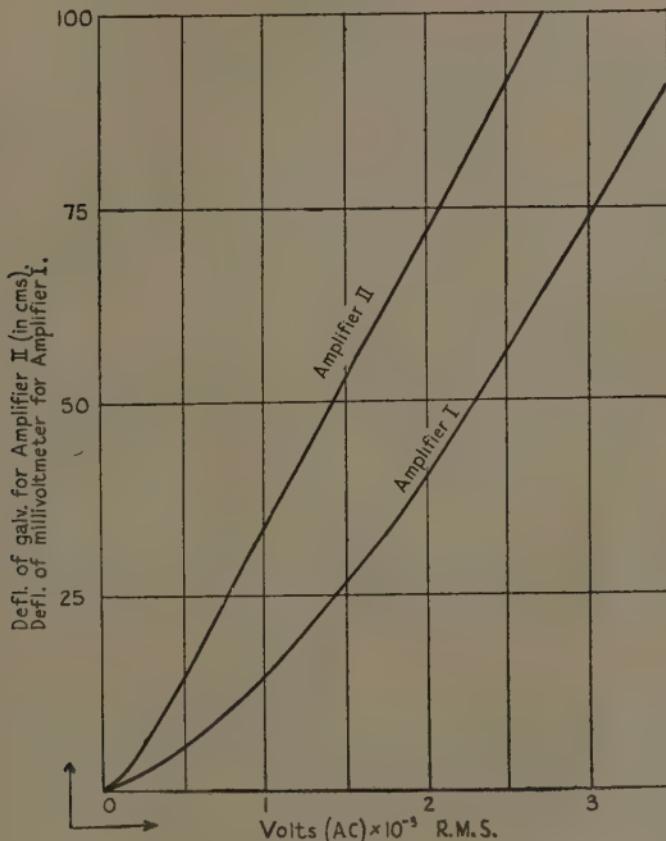
## I. Disturbance of Supraconductivity by a Magnetic Field.

A. *Disturbance in Longitudinal Field.*

The arrangement used had the axes of primary and secondary parallel to the axis of the cylinder. The specimen was kept at constant temperature and the amplitude of the A.C. field equal to 3·8 gauss, while the D.C. field was increased from zero to above the threshold

value and then decreased again to zero. This increase and subsequent decrease of the D.C. field was repeated several times, and it was found in all cases that the values obtained for the induced e.m.f. in the secondary were quite reproducible.

A number of experiments at different temperatures was made, and a typical curve showing the induced e.m.f. in the secondary against D.C. field strength taken at 3·444° K. is given in fig. 3.

Fig. 3.
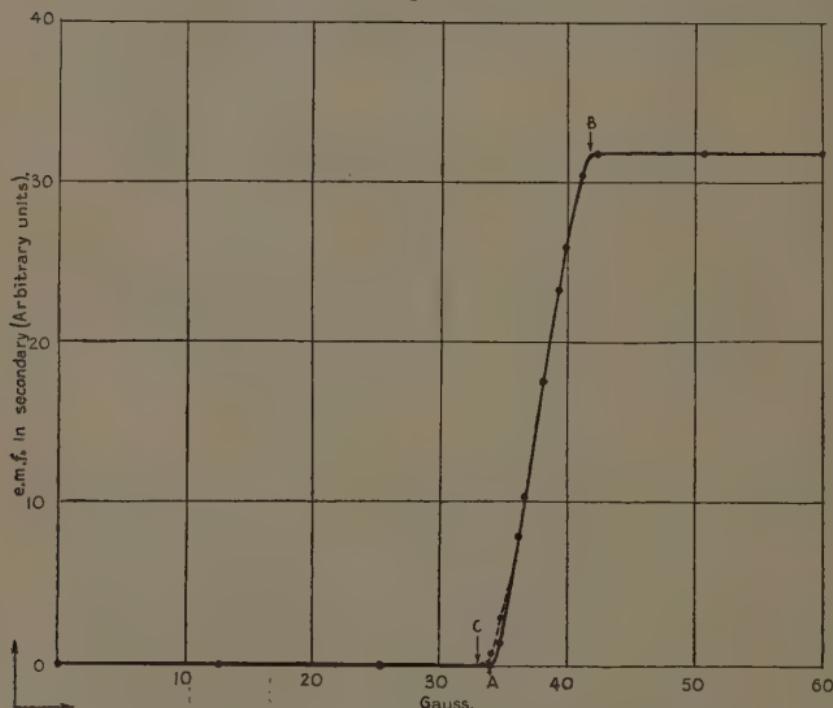
From this curve it will be seen that, when increasing the field, until the field value A is reached the induced e.m.f. in the secondary is zero. At A the e.m.f. suddenly increases, showing that here the transition from the supraconductive to the normal state begins. The e.m.f. in the secondary continued to increase with increasing field up to the field B, indicating that the transition occurs over a finite interval of field strength.

As the field was increased above the value B the induced e.m.f. remained constant, showing that at B the transition

was completed and that the resistance of tin in the normal state at this temperature is unaffected by fields of this magnitude.

On decreasing the field the induced e.m.f. in the secondary started suddenly to decrease at B and followed the curve for increasing fields until near A. Here a slight, but reproducible, hysteresis occurred (shown by the dotted curve), and it was not until the field value C was reached that the specimen came again completely into the supraconductive state.

The finite range of transition found in this experiment is not due to the specimen having a finite demagnetisation factor. For in applying magnetic fields longitudinally the demagnetisation factor of the hollow cylinder (after the field was increased above B for the first time) was sensibly equal to zero. The breadth of the transition is due, however, to the superposition of the A.C. field on the D.C. field ; and, as is to be expected, the range of transition is exactly equal to twice the amplitude of the superimposed A.C. field. This was found to be true in a number of similar experiments using A.C. fields of various magnitudes.

In consequence the true threshold value which would be obtained if the A.C. field were reduced to zero (*i. e.*, if the range of transition was made infinitely small) is the middle of the transition range. In the curve given for $3 \cdot 444°$ K. the threshold field so obtained is 38 gauss *, which is in agreement with measurements made with other methods [8].

B. *Disturbance in Transverse Field.*

In destroying supraconductivity by applying transverse magnetic fields at constant temperature three different methods 1, 2, 3, were employed.

(1) Again the arrangement of experiment A was employed with the axes of the primary and secondary parallel to the axis of the cylinder. The specimen was kept at constant temperature and the amplitude of the A.C. field equal to $3 \cdot 8$ gauss, while the transverse field was repeatedly increased from zero to above the threshold value and decreased to zero. A typical curve for the

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* These longitudinal threshold values were obtained at many different temperatures with this method, not only for tin, but also for lead and mercury and will be described in detail in a later publication.

e.m.f. induced in the secondary against the field strength at 3·525° K. is given in fig. 4.

From the curve it will be seen that the induced e.m.f. in the secondary was zero until the value of the field A′ was reached. Above A′ the e.m.f. increased (not, however, linearly) until it reached the field B′, above which it remained constant, indicating that at B′ the transition was complete. When the field was lowered again the

Fig. 4.
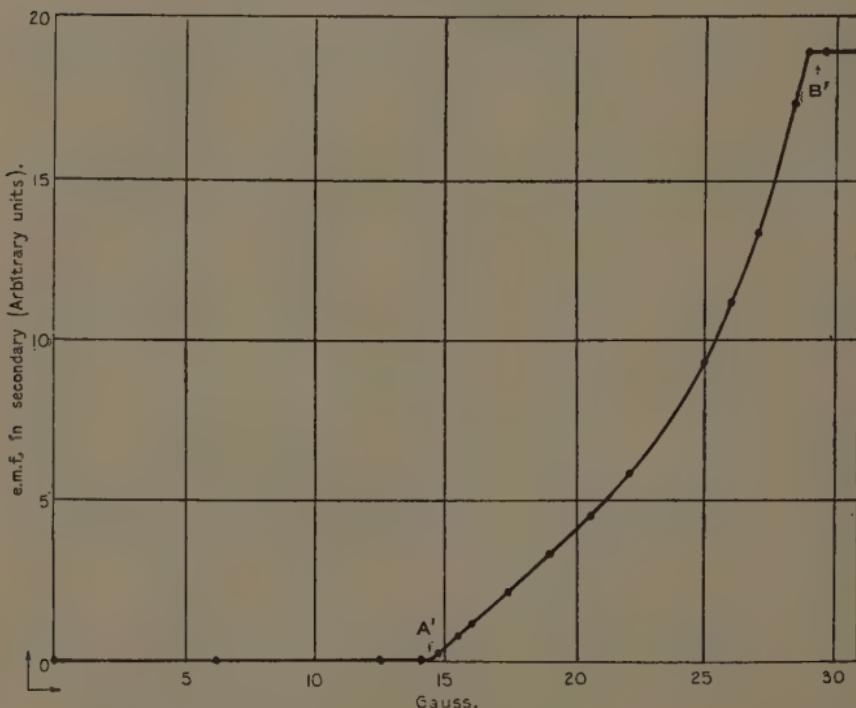
induced e.m.f. reversed the path it followed on increase of field. The experiment was repeated after magnetic flux had been " frozen in " in a perpendicular transverse direction by applying a transverse field higher than the threshold value and reducing it to zero again. In this case, too, the same curve was traced for increasing and decreasing fields.

(2) The arrangement employed had the axes of the primary and secondary perpendicular to the axis of the cylinder, and the D.C. field was applied in a direction

perpendicular to the direction of the A.C. field. Fig. 5 gives a diagrammatic sketch of the arrangement.

The specimen was kept at a constant temperature and the amplitude of the A.C. field equal to 0·42 gauss,

Fig. 5.

Direction of DC Field

Axis of
Primary.
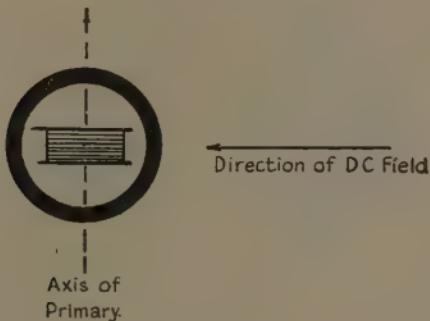
Fig. 6.
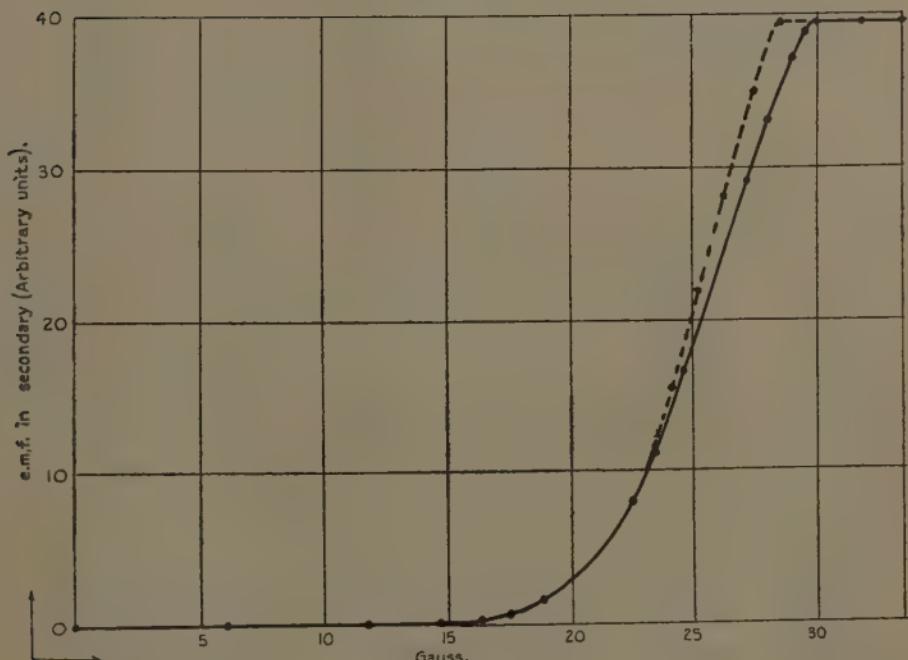
while the transverse field was increased from zero to above the threshold value and then decreased to zero. A typical curve of the results taken at 3·515° K. showing the relation between the induced e.m.f. in the secondary and the D.C. field strength is given in fig. 6.

(3) The same arrangement was employed and the experiment carried out exactly as in B (2) with the exception that the D.C. field was now orientated parallel to the A.C. field (see fig. 7).

Fig. 7.

Axis of
Primary

Direction of
D C Field
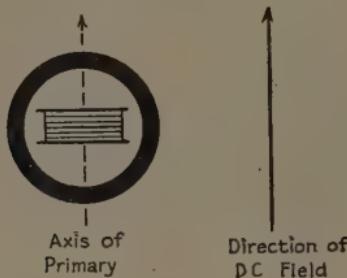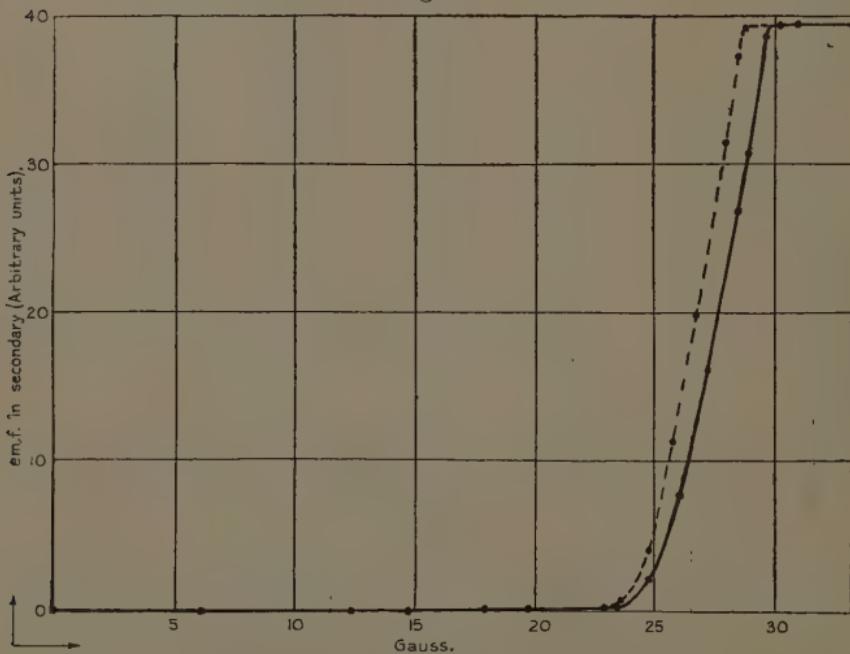
Fig. 8.


The results of this arrangement are indicated in the curve shown in fig. 8, which was made at the same temperature of 3·515° K. and with the same A.C. field of amplitude 0·42 gauss as were used for the experiment B (2).

It will be clearly seen from experiments B (1) and B (2) (see figs. 4 and 6) that the transition began at a value of the field equal to half the field value at which the

transition was completed., In fig. 4, for example, A' and
B', which mark the beginning and ending of the transition,
correspond to fields of 14·4 and 28·8 gauss. This was
also observed in experiment B (3), although it is seen less
clearly from the curve (fig. 8). In this case, when the
increasing field reached a value of 14·7 gauss, a quite
definite though small deflexion was observed on the
galvanometer, this value of the field being exactly half
the value (29·4 gauss) at which the transition was com-
pleted.

In the experiments B (2) and B (3) the curves for
decreasing fields show hysteresis effects, whereas in experi-
ment B (1) no hysteresis was observed. When decreasing
the field in experiments B (2) and B (3) the induced e.m.f.
in the secondary did not begin to decrease until a value
of the field was reached well below that corresponding
to the completion of the transition in increasing fields.
This is shown by the dotted curves in figs. 6 and 8. The
induced e.m.f. reached zero, however, at the same value
of the field at which it first appeared in increasing fields.
These hysteresis effects were found to be quite repro-
ducible.

Discussion.

The disturbance of supraconductivity in transverse
fields is necessarily a far more complicated phenomenon
than that occurring in longitudinal fields. As the experi-
ments made with longitudinal fields show (provided that
the A.C. field is made negligibly small), the destruction
of supraconductivity occurs suddenly at the threshold
value (H_c) at which the shielding persistent currents
over the whole specimen break down. In transverse
fields, however, the transition from the supraconductive
to the normal state begins, *i. e.*, the first leaks in the
supraconductive shield occur at exactly $0·5\,H_c$ (where
H_c is the field at which the transition is completed),
where supraconductivity, according to electro-dynamical
theory, has first to break down on the surface of the
cylinder. The values of H_c found in these transverse
experiments (*e. g*, 29·4 gauss at 3·515° K.) are in agree-
ment with the equilibrium values observed in induction
experiments [8].

From these experiments with transverse fields, however,
an idea can be obtained about the formation of the

normal and supraconductive regions which must occur during the transition. Although a detailed picture of the shape of the individual regions cannot be realised from the results, it is evident that the shielding currents have quite different values at the same strength of the D.C. field, dependent on whether the A.C. field is parallel or perpendicular to the D.C. field.

Whereas in the case when the D.C. field is perpendicular to the A.C. field, the electromagnetic changes observed inside the cylinder increase steadily in intensity from zero, when the external magnetic field is equal to $0.5\ H_c$, to their full value at H_c ; in the case where the A.C. field is parallel to the D.C. field these electromagnetic changes do not begin to rise appreciably until the D.C. field has reached a value $0.75\ H_c$. (As pointed out before, however, a very slight effect was observed at $0.5\ H_c$.) And over the entire range of field values $0.5\ H_c$ to H_c the intensity of the electromagnetic changes occurring inside the cylinder is greater in the former arrangement than in the latter.

The supraconductive and normal regions therefore must be disposed so that persistent shielding currents offer a smaller effective cross-section to A.C. fields perpendicular to the D.C. field than to A.C. fields parallel to the D.C. field.

II. Time-Effects.

Mendelssohn and Pontius have shown that the changes of induction with time that occur in the transition region, when one of the variables of state (temperature or magnetic field) is changed abruptly, vary greatly with the shape of the specimen [4]. This fact strongly supports the assumption that these time-effects are due to a slow expansion or contraction of macroscopic supraconductive regions. It could be deduced from theoretical considerations by London [2] and Peierls [3] that no co-existence of macroscopic supraconductive and normal regions will occur in bodies of very regular shape (spheres, ellipsoids). However, it seems necessary to assume such a co-existence in bodies of irregular shape (cylinders with sharp edges, etc.), and a hollow cylinder in a transverse magnetic field represents such a complicated shape in which we have to expect supraconductive and normal regions in equi-

librium, as soon as lines of force have penetrated the specimen for the first time at $0.5\ H_c$.

The arrangement described in this paper offers therefore a useful method to investigate these time-effects in detail.

Method.

Employing the long hollow tin cylinders used in the experiments just described, time-effects were observed in the following manner : the cylinder was placed in a transverse magnetic field of value between $0.5\ H_c$ and H_c, so that the cylinder was split up into macroscopic regions of supraconductive and non-supraconductive material in equilibrium (as mentioned previously). Then the field was rapidly turned through an angle of $90°$ and the time required for these regions to reorientate themselves in the new direction demanded by the new direction of the transverse magnetic field was observed. This observation was made possible by having, as in the previous experiments, a small A.C. magnetic field round the specimen and observing as before the e.m.f. induced in a secondary coil situated inside the cylinder. The time-effects were observed with both longitudinal and transverse secondary coils and were found to be independent of the direction of the secondary.

Results.

The curve given in fig. 9 is a typical one (using a longitudinal secondary) showing the relation between induced e.m.f. in the secondary and time. The zero of time is taken at the instant the D.C. coil was turned through $90°$. The curve was made at $3.525°$ K. with an A.C. field of amplitude 3.8 gauss and a D.C. field of value $0.74\ H_c$.

It will be seen that immediately after the direction of the D.C. field was changed the e.m.f. in the secondary suddenly decreased and came back gradually with time to its original value, indicating that the distribution of supraconductive and normal regions reached in time its original order with respect to the external field.

The time-effects observed in this way were independent of the strength of the A.C. field, and, as mentioned previously, independent of the orientation of the secondary. Also it was immaterial whether the D.C. field was changed

from position 1 to position 2, or if it was changed in the reverse direction from position 2 to position 1.

The time-effect varied strongly, however, with the strength of the D.C. magnetic field. Whereas in the experiment at 0.74 H_c represented in fig. 9 equilibrium was reached after about 20 seconds, no change of mutual inductance was observed at 0.825 H_c after 10 seconds, and at 0.915 H_c after 5 seconds. Longer times were needed at lower fields : 40 secs. at 0.65 H_c, 60 secs. at

Fig. 9.
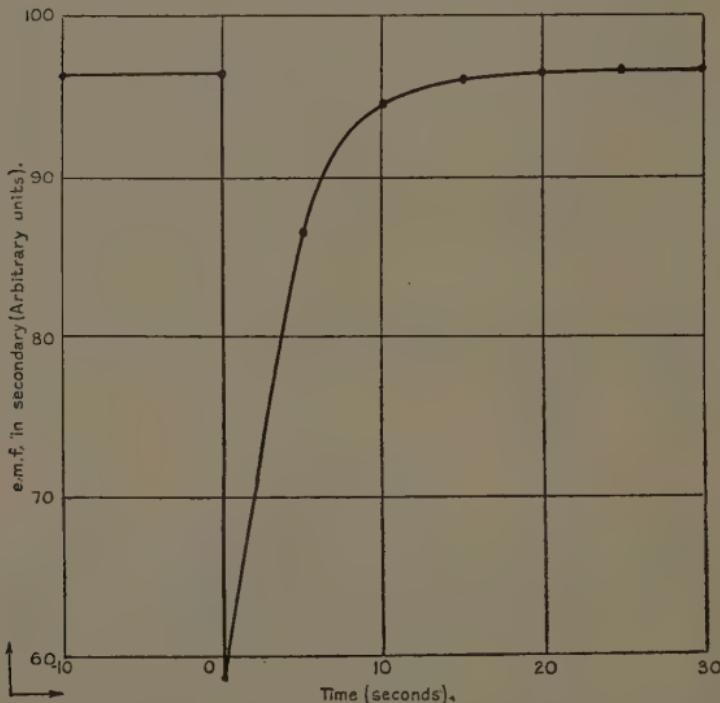
0.565 H_c, and 180 secs. at 0.53 H. All the curves taken at different strengths of the D.C. field were exactly the same in shape as the one given in fig. 9, although the magnitude of the jump was dependent on the strength of the D.C. field used, and was greatest at 0.75 H_c. The curve given in fig. 10 shows the relation between the time taken for the e.m.f. induced in the secondary to reach half its original value and the strength of the D.C. field expressed in fractions of H_c. Although the time-effect curves were not exactly exponential functions they are

all similar, and, consequently, this half-value period gives a suitable indication of the time-effect, especially as the total time taken for the e.m.f. in the secondary to reach its original value is less clearly defined.

From the curve in fig. 10 it will be seen that the time taken for rearranging the distribution of the supra-conductive and normal regions in the cylinder tends to infinity as the field tends to $0.5\ H_c$. This offers an

Fig. 10.
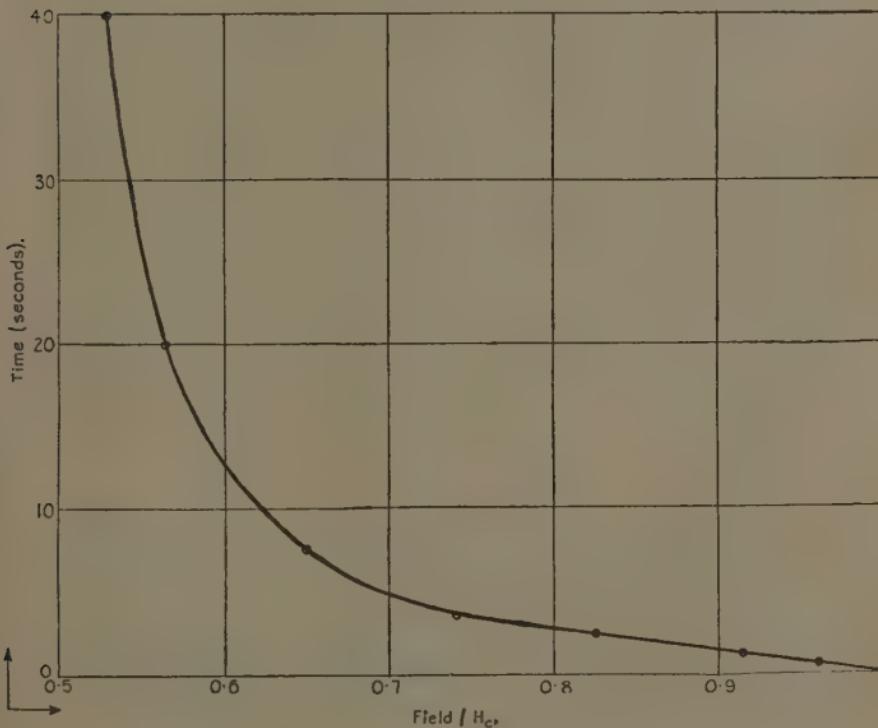
explanation of the experiment of Tuyn and Onnes [9] with a hollow lead sphere, in which, owing to the small field produced by the outer lead ring, no redistributon of the persistent currents (*i. e.*, no redistribution of the supraconductive and normal regions) could occur in finite time. It will be seen, further, that as the field tends to the value H the time-effect tends to zero.

In order to decide whether these time-effects were due to the fact that under the conditions used the cylinder was split up into macroscopic supraconductive and non-supraconductive regions or whether it was due to some

other cause, a separate experiment was carried out. For this a solid rod made from the same tin was used with a secondary wound on its exterior so that the axis of the secondary coincided with the axis of the rod. According to the treatment of London and Peierls no co-existence of supraconductive and normal regions will occur in a solid cylindrical rod in a transverse magnetic field ; but as soon as the external field reaches $0.5 \, H_c$ the rod will pass into the intermediate state. The same procedure as just described was adopted, keeping the specimen at a constant temperature of $3.525°$ K. in an A.C. field of amplitude 3.8 gauss : and a constant transverse magnetic field of value $0.58 \, H_c$ was applied. In this case no time-effects were observed (unless they had a value of less than 0.5 secs., which was the minimum limit observable). This experiment shows that the time-effects observed in the previous experiments were dependent on the conditions obtaining in a long hollow cylinder, and not due to any other possible causes (*e. g.*, magneto-caloric effects, etc.). Also it shows that the intermediate supraconductive state, in which the solid existed under the given conditions, showed no relaxation phenomena greater than 0.5 secs. This confirms the experiments previously made on this subject by Mendelssohn and Pontius [4] *.

Acknowledgments.

In conclusion I wish to thank Dr. K. Mendelssohn, who suggested these experiments, for help and advice in various phases of the work, and to Professor F. A. Lindemann, F.R.S., for the interest he has taken in the research and for extending to me the facilities of the Clarendon Laboratory. Further, I should like to thank Mr. R. B. Pontius for his kind help during the experiments.

* *Added in proof.*—Very recently Keesom and van Laer (*Physica*, iv. p. 447, 1937) have observed time effects after heating an ellipsoidal block of tin in the transition region. It has to be remembered, however, that a *local* heating will bring momentarily a part of the specimen into the normal state, and the subsequent rearrangement into the intermediate state will require some time. This time will *not* depend only on the attainment of thermal equilibrium, but will be of far greater order (see, for example, K. Mendelssohn, Proc. Roy. Soc. A, clv. p. 568, 1936). This explanation is also supported by the fact that no time effects were observed when the external magnetic field was altered, as in the latter case the variable of state (H) was changed homogeneously throughout the whole specimen. But it has, of course, to be left to further experiments to test the validity of this assumption. I am indebted to Dr. K. Mendelssohn for pointing out this possible explanation.

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(5) The possibility of this was suggested by J. C. McLennan (' Nature,' Suppl., Dec. 10th, 1932) and has recently been investigated in the infra-red region by J. G. Daunt, T. C. Keeley, and K. Mendelssohn, Phil. Mag. ser. 7, xxiii. p. 264 (1937), and by E. Hirschlaff, Proc. Camb. Phil. Soc. xxxiii. p. 140 (1937).

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(9) W. Tuyn and H. Kammerlingh Onnes, Leiden Comm., Suppl. 50 *a* and 198 (1929).

XXXVIII. *The Magneto-Optical Dispersion of Organic Liquids in the Ultra-violet Region of the Spectrum.*— Part XI. *The Magneto-Optical Dispersion of Methyl Malonate, Isopropyl Propionate, Isopropyl Butyrate, Ethyl Isobutyrate, and Tertiary-Butyl Alcohol. By* C. Joyce Marsden, *Ph.D., and* Prof. E. J. Evans, *D.Sc. (Physics Department, University College of Swansea)* *.

Introduction.

IN the present investigation the magneto-optical rotations and the refractive indices of the five organic liquids have been measured for wave-lengths in the violet and ultra-violet regions of the spectrum. The experimental methods [1] employed in the measurements have been already described in detail, and in this paper it is only necessary to record the experimental results, which

* Communicated by Prof. E. J. Evans.

will be considered in relation to Larmor's [2] theory of magnetic rotation. According to this theory the value of Verdet's constant δ for a wave-length λ is given by the expression

$$\delta \doteqdot \frac{e}{2mc^2} \lambda \cdot \frac{dn}{d\lambda}, \quad . \quad . \quad . \quad . \quad (1)$$

where n is the refractive index, e/m is the ratio of the charge to the mass of the resonators, and c is the velocity of light. In the above expression the charge e is measured in electrostatic units, and the magnetic field in electromagnetic units.

If the ordinary dispersion of each liquid is represented by an equation of the type

$$n^2 - 1 = b_0 + \frac{b_1}{\lambda^2 - \lambda_1^2} + . \quad .., \quad . \quad . \quad . \quad (2)$$

and if the magneto-optical dispersion over the range of wave-length investigated is controlled by one absorption band in the Schumann-Lyman region, it can be shown that

$$n\delta\lambda^2 \doteqdot K \left(\frac{\lambda^2}{\lambda^2 - \lambda_1^2} \right)^2, \quad . \quad . \quad . \quad . \quad (3)$$

where K is a constant and λ_1 is the wave-length of the absorption band.

In addition, the value of e/m deduced from the above three equations is given by the expression

$$\frac{2KC^2}{D_1}.$$

All the liquids used in this investigation were obtained from Dr. Schuchardt of Görlitz. The methyl malonate, which had been twice distilled *in vacuo*, was employed without further purification. The pure tertiary-butyl alcohol solidifies at $25 \cdot 5°$ C, and can thus be separated from any water the alcohol may have absorbed. Each of the remaining three liquids were subjected to a process of fractional distillation, and the fraction which distilled over near the accepted boiling-point was employed in the determinations of the refractive indices and magneto-optical rotations. The isopropyl propionate distilled

over between 109·3° and 109·7° C. at normal pressure, the
isopropyl butyrate between 130·5° and 131° C. at 775 mm.
pressure, and the ethyl isobutyrate between 109·6° and
109·8° C. at a pressure of 753 mm.

The refractive indices of methyl malonate, isopropyl
propionate, isopropyl butyrate, ethyl isobutyrate, and
tertiary-butyl alcohol for wave-lengths between ·6678
and ·4472 μ were determined visually at 22·9°, 14°, 17·3°,
18°, and 27° C. respectively, and their refractive indices
in the visible and ultra-violet regions of the spectrum were
measured photographically at 17·5°, 22°, 25°, 18°, and 27°C.
respectively. The measurements of the magneto-optical
rotations were not generally carried out at the above
temperatures, and the refractive indices were therefore
corrected to the temperatures at which the magneto-
optical rotations were measured by making use of the
values of the temperature coefficients of the refractive
indices of the liquids. It was found that the refractive
indices of methyl malonate, isopropyl propionate, iso-
propyl butyrate, ethyl isobutyrate, and tertiary-butyl
alcohol diminished by about $\cdot0004_4$, $\cdot0004_7$, 0004_8, $\cdot0004_8$,
and $\cdot0005_4$ respectively per degree C. rise of temperature.

Experimental Results.

Refractive Indices.

Table I. (a).

Visual Determinations.

| Wave-length in microns.	Refractive Indices.				
	Methyl malonate at 14·7° C.	Isopropyl propionate at 9·5° C.	Isopropyl butyrate at 11·4° C.	Ethyl isobutyrate at 11·0° C.	Tertiary-butyl alcohol. at 27·0° C.
·7065	—	$1\cdot3885_4$	$1\cdot3947_6$	—	—
·6678	$1\cdot4136_1$	$1\cdot3892_9$	$1\cdot3955_7$	$1\cdot3918_0$	$1\cdot3819_6$
·5876	$1\cdot4160_6$	$1\cdot3916_3$	$1\cdot3978_6$	$1\cdot3941_3$	$1\cdot3842_6$
·5016	$1\cdot4201_3$	$1\cdot3953_0$	$1\cdot4017_5$	$1\cdot3981_3$	$1\cdot3880_2$
·4922	$1\cdot4207_3$	$1\cdot3958_5$	$1\cdot4022_7$	$1\cdot3985_8$	$1\cdot3885_3$
·4713	$1\cdot4221_6$	$1\cdot3971_4$	$1\cdot4035_8$	$1\cdot3998_7$	$1\cdot3898_7$
·4472	$1\cdot4240_6$	$1\cdot3989_8$	$1\cdot4054_1$	$1\cdot4016_1$	$1\cdot3916_9$

TABLE I. (b).—Photographic Determinations.

Methyl malonate at 14·7° C.		Isopropyl propionate at 9·5° C.		Isopropyl butyrate at 11·4° C.		Ethyl isobutyrate at 11·0° C.		Tertiary-butyl alcohol at 27·0° C.	
Wave-length in microns.	Refractive index.	Wave-length in microns.	Refractive index.	Wave-length in microns.	Refractive index.	Wave-length in microns.	Refractive index.	Wave-length in microns.	Refractive index.
·4675	$1·4224_4$	·4705	$1·3972_0$	·5782	$1·3981_7$	·4540	$1·4011_1$	·4767	$1·3894_5$
·4587	$1·4231_3$	·4698	$1·3972_2$	·5106	$1·4012_5$	·4275	$1·4034_6$	·4378	$1·3923_5$
·4531	$1·4235_8$	·4675	$1·3974_0$	·4705	$1·4036_1$	·4023	$1·4062_7$	·4249	$1·3936_4$
·4416	$1·4245_5$	·4651	$1·3976_0$	·4540	$1·4048_2$	·3881	$1·4080_1$	·4023	$1·3960_4$
·4378	$1·4249_3$	·4540	$1·3984_2$	·4275	$1·4072_2$	·3861	$1·4082_4$	·3825	$1·3986_0$
·4249	$1·4262_4$	·4531	$1·3985_4$	·4063	$1·4094_3$	·3689	$1·4106_7$	·3656	$1·4009_8$
·4178	$1·4269_6$	·4416	$1·3995_2$	·3881	$1·4114_5$	·3624	$1·4117_7$	·3534	$1·4031_2$
·4063	$1·4283_3$	·4260	$1·4009_6$	·3825	$1·4122_1$	·3601	$1·4121_4$	·3512	$1·4034_2$
·3821	$1·4315_4$	·4123	$1·4022_9$	·3599	$1·4157_2$	·3484	$1·4143_5$	·3489	$1·4039_6$
·3656	$1·4342_6$	·3974	$1·4040_1$	·3458	$1·4182_8$	·3449	$1·4150_2$	·3474	$1·4041_2$
·3624	$1·4347_9$	·3861	$1·4054_2$	·3319	$1·4211_2$	·3337	$1·4173_4$	·3414	$1·4053_2$
·3602	$1·4350_5$	·3700	$1·4078_1$	·3176	$1·4246_3$	·3208	$1·4203_0$	·3365	$1·4063_2$
·3405	$1·4390_5$	·3602	$1·4093_7$	·3109	$1·4264_1$	·3142	$1·4221_5$	·3308	$1·4075_5$
·3382	$1·4395_1$	·3498	$1·4111_6$	·3074	$1·4274_0$	·3100	$1·4232_0$	·3274	$1·4083_2$
·3338	$1·4405_9$	·3450	$1·4121_2$	·3036	$1·4286_3$	·3064	$1·4244_2$	·3247	$1·4089_5$
·3274	$1·4421_2$	·3422	$1·4127_9$	·2961	$1·4312_5$	·2961	$1·4281_0$	·3176	$1·4107_0$
·3247	$1·4429_1$	·3414	$1·4129_8$	·2883	$1·4339_8$	·2883	$1·4305_0$	·3152	$1·4113_2$
·3208	$1·4439_4$	·3274	$1·4159_8$	·2824	$1·4361_0$	·2824	$1·4327_2$	·3147	$1·4115_7$
·3194	$1·4442_3$	·3170	$1·4185_2$			·2766	$1·4352_1$	·3074	$1·4135_6$
·3142	$1·4457_1$	·3142	$1·4192_6$					·3064	$1·4138_3$
·3109	$1·4467_4$	·3064	$1·4216_0$					·2961	$1·4170_4$
·3094	$1·4472_4$	·2924	$1·4261_6$					·2883	$1·4199_4$
·3064	$1·4481_4$	·2825	$1·4299_6$					·2825	$1·4219_9$
·2997	$1·4503_9$								

The refractive indices of the five liquids are plotted in Graph 1.

Graph 1.
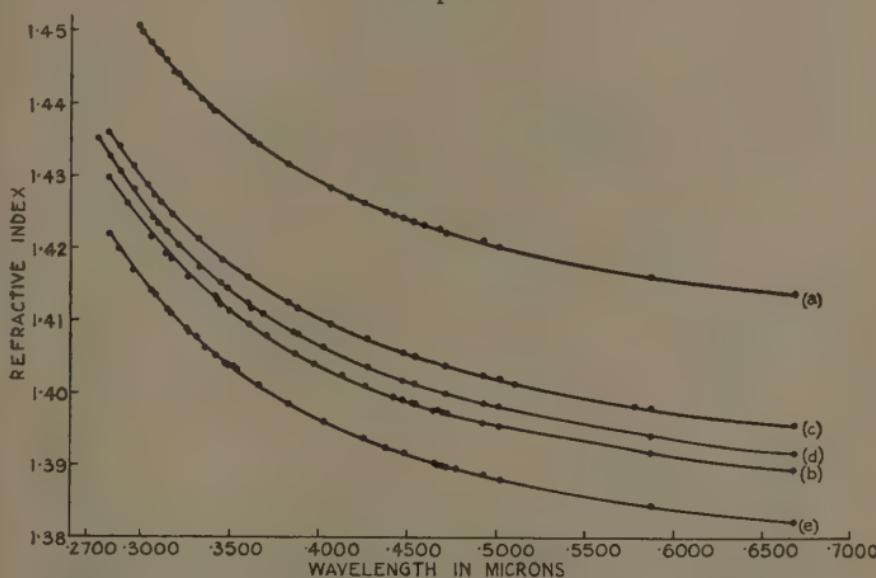
Refractive indices of (a) Methyl malonate at $14.7°$ C. : (b) Isopropyl propionate at $9.5°$ C.; (c) Isopropyl butyrate at $11.4°$ C.; (d) Ethyl isobutyrate at $11°$ C.; (e) Tertiary-butyl alcohol at $27°$ C.

Magneto-optical Dispersion.

Methyl Malonate.

The experimental results are given in Table II. (a) and in Graph 2 (a).

TABLE II. (a).

Values of Verdet's Constant at various wave-lengths for Methyl Malonate at $14.7°$ C.

Wave-length in microns.	Verdet's constant in min./cm. gauss.	Wave-length in microns.	Verdet's constant in min./cm. gauss.
·4722	·0177$_0$	·3940	·0265$_5$
·4593	·0188$_0$	·3871	·0276$_5$
·4474	·0199$_2$	·3805	·0287$_7$
·4368	·0210$_2$	·3745	·0298$_8$
·4271	·0221$_3$	·3688	·0309$_8$
·4178	·0232$_3$	·3632	·0321$_0$
·4092	·0243$_3$	·3580	·0332$_0$
·4014	·0254$_5$	·3532	·0343$_1$

: It was found that the above results could be represented by the formula

$$n\delta\lambda^2 = K \left(\frac{\lambda^2}{\lambda^2 - \lambda_1^2} \right)^2 ,$$

and the values of the constants K and λ_1 were calculated from the values of n and δ given in Table II. (b).

Graph 2.
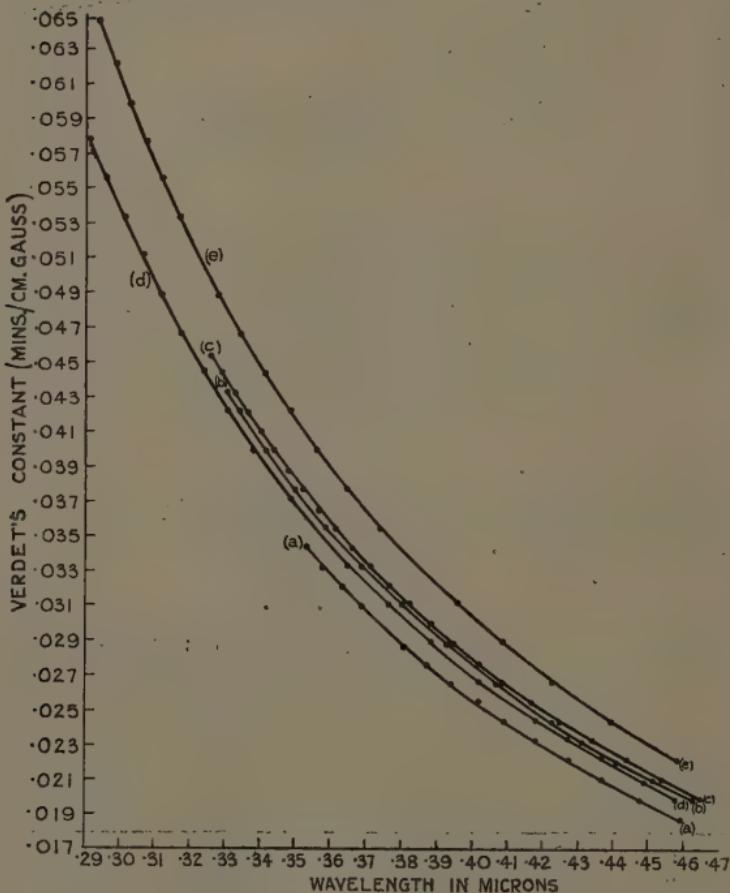
Magnetic rotary dispersion of (a) Methyl malonate at 14·7° C.; (b) Isopropyl propionate at 9·5° C.; (c) Isopropyl butyrate at 11·4° C.; (d) Ethyl isobutyrate at 11° C.; (e) Tertiary-butyl alcohol at 27° C.

TABLE II. (b).

	Wave-length λ (microns).	Refractive index n.	Verdet's constant δ.
(a)	$\cdot 4474_5$	$1 \cdot 4240_5$	$\cdot 0199_2$
(b)	$\cdot 3940$	$1 \cdot 4299_1$	$\cdot 0265_5$
(c)	$\cdot 3532$	$1 \cdot 4365_0$	$\cdot 0343_1$

From (a) and (b) $\lambda_1 = \cdot 1093_5 \, \mu$ and $K = 5 \cdot 020_1 \times 10^{-3}$.
,, (a) and (c) $\lambda_1 = \cdot 1101_5 \, \mu$ and $K = 5 \cdot 010_7 \times 10^{-3}$.
,, (b) and (c) $\lambda_1 = \cdot 1107_5 \, \mu$ and $K = 4 \cdot 999_0 \times 10^{-3}$.

The mean value of $\lambda_1 = \cdot 1101 \, \mu$ and of $K = 5 \cdot 01 \times 10^{-3}$, and hence the equation representing the magneto-optical dispersion of methyl malonate at $14 \cdot 7^\circ$ C. within the range $\cdot 4722 \, \mu$ to $\cdot 3532 \, \mu$, is

$$n\delta = 5 \cdot 01_0 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (\cdot 1101)^2\}^2} \cdot$$

From this equation were calculated values of Verdet's constant at wave-lengths for which the experimental values had been determined, and the comparison between the calculated and observed values is given in Table II. (c).

TABLE II. (c).

Wave-length λ in microns.	Verdet's constant δ (observed).	Verdet's constant δ (calculated).
$\cdot 4722$	$\cdot 0177_0$	$\cdot 0176_7$
$\cdot 4368$	$\cdot 0210_2$	$\cdot 0210_1$
$\cdot 4178$	$\cdot 0232_3$	$\cdot 0232_2$
$\cdot 4014$	$\cdot 0254_5$	$\cdot 0254_4$
$\cdot 3805$	$\cdot 0287_7$	$\cdot 0287_9$
$\cdot 3688$	$\cdot 0309_8$	$\cdot 0309_7$
$\cdot 3580$	$\cdot 0332_0$	$\cdot 0332_2$

The value of δ for sodium light calculated from the magneto-optical dispersion equation for methyl malonate at $14 \cdot 7^\circ$ C. is $\cdot 0109_4$. Perkin[3] has found the specific rotation of this liquid to be $0 \cdot 8339$ at $17 \cdot 1^\circ$ C., and after correction to $14 \cdot 7^\circ$ C. the specific rotation becomes $0 \cdot 8355$. The value of Verdet's constant of methyl malonate for sodium light deduced from Perkin's result

(taking Verdet's constant of water at $14 \cdot 7°$ C. for this wave-length to be $\cdot01309_4$) is $\cdot0109_5$.

Isopropyl Propionate.

The experimental results are given in Table III. (*a*) and in Graph 2 (*b*).

TABLE III. (*a*).

Values of Verdet's Constant at various wave-lengths for Isopropyl Propionate at $9 \cdot 5°$ C.

Wave-length in microns.	Verdet's constant in min./cm. gauss.	Wave-length in microns.	Verdet's constant in min./cm. gauss.
$\cdot4625$	$\cdot0199_4$	$\cdot3800$	$\cdot0310_2$
$\cdot4512$	$\cdot0210_4$	$\cdot3689$	$\cdot0332_4$
$\cdot4409$	$\cdot0221_6$	$\cdot3589$	$\cdot0354_5$
$\cdot4314$	$\cdot0232_6$	$\cdot3499$	$\cdot0376_7$
$\cdot4226$	$\cdot0243_7$	$\cdot3416$	$\cdot0398_9$
$\cdot4068$	$\cdot0265_6$	$\cdot3340$	$\cdot0421_0$
$\cdot3927$	$\cdot0288_0$	$\cdot3304$	$\cdot0432_2$

The values of the constants K and λ_1 of the equation

$$n\delta\lambda^2 = K\left(\frac{\lambda^2}{\lambda^2 - \lambda_1{}^2}\right)^2,$$

which represents the experimental results for isopropyl propionate, were obtained from the data collected in Table III. (*b*).

TABLE III. (*b*).

	Wave-length λ (microns).	Refractive index *n*.	Verdet's constant δ.
(*a*)	$\cdot4512$	$1 \cdot 3986_5$	$\cdot0210_4$
(*b*)	$\cdot3800$	$1 \cdot 4063_2$	$\cdot0310_2$
(*c*)	$\cdot3304$	$1 \cdot 4152_5$	$\cdot0432_2$

From (*a*) and (*b*) $\lambda_1 = \cdot1077_5 \, \mu$ and $K = 5 \cdot 327_1 \times 10^{-3}$.

,,   (*a*) and (*c*) $\lambda_1 = \cdot1082_2 \, \mu$ and $K = 5 \cdot 321_5 \times 10^{-3}$.

,,   (*b*) and (*c*) $\lambda_1 = \cdot1086_6 \, \mu$ and $K = 5 \cdot 312_2 \times 10^{-3}$.

The mean values of λ_1 and K are $\cdot1082\,\mu$ and $5\cdot32_0\times10^{-3}$ respectively, and, therefore, the equation giving the magneto-optical dispersion of isopropyl propionate at $9\cdot5°$ C. in the region of the spectrum extending from $\cdot4625\,\mu$ to $\cdot3304\,\mu$ is

$$n\delta = 5\cdot32_0\times10^{-3}\frac{\lambda^2}{\{\lambda^2-(\cdot1082)^2\}^2}$$

This equation was used to calculate the values of Verdet's constant at wave-lengths for which experimental values had been obtained, and a comparison between the calculated and observed values is given in Table III. (*c*).

TABLE III. (*c*).

Wave-length λ (microns).	Verdet's constant δ (observed).	Verdet's constant δ (calculated).
$\cdot4625$	$\cdot0199_4$	$\cdot0199_1$
$\cdot4314$	$\cdot0232_6$	$\cdot0232_6$
$\cdot4226$	$\cdot0243_7$	$\cdot0243_5$
$\cdot3927$	$\cdot0288_0$	$\cdot0287_6$
$\cdot3689$	$\cdot0332_4$	$\cdot0332_4$
$\cdot3589$	$\cdot0354_5$	$\cdot0354_5$
$\cdot3499$	$\cdot0376_7$	$\cdot0376_5$
$\cdot3416$	$\cdot0398_9$	$\cdot0398_7$
$\cdot3340$	$\cdot0421_0$	$\cdot0420_9$

The value of Verdet's constant of isopropyl propionate at $9\cdot5°$ C. for sodium light calculated from the magneto-optical dispersion equation is $\cdot0117_9$. According to Perkin [4] the specific rotation of isopropyl propionate at $11°$ C. for sodium light is $\cdot8966$, and correcting for temperature the specific rotation at $9\cdot5°$ C. becomes $\cdot8979$. Taking the value of Verdet's constant of water at $9\cdot5°$ C. for sodium light to be $\cdot01310_4$, the value of Verdet's constant of isopropyl propionate at $9\cdot5°$ C., deduced from Perkin's result, is $\cdot0117_7$.

Isopropyl Butyrate.

The experimental results are given in Table IV. (*a*) and in Graph 2 (*c*).

TABLE IV. (a).

Values of Verdet's Constant at various wave-lengths for Isopropyl Butyrate at 11·4° C.

Wave-length in microns.	Verdet's constant in min./cm. gauss.	Wave-length in microns.	Verdet's constant in min./cm. gauss.
·4647	·0199$_4$	·3712	·0332$_5$
·4537	·0210$_5$	·3662	·0343$_6$
·4434	·0221$_8$	·3612	·0354$_5$
·4337	·0232$_8$	·3565	·0365$_8$
·4249	·0243$_8$	·3520	·0376$_8$
·4167	·0254$_6$	·3478	·0387$_9$
·4088	·0266$_0$	·3439	·0398$_9$
·4016	·0277$_0$	·3399	·0410$_0$
·3948	·0288$_1$	·3361	·0421$_1$
·3883	·0299$_3$	·3325	·0432$_3$
·3823	·0310$_3$	·3289	·0443$_3$
·3766	·0321$_4$	·3257	·0454$_4$

The constants K and λ_1 of the magneto-optical dispersion equation of isopropyl butyrate were calculated from the data given in Table IV. (b).

TABLE IV. (b).

	Wave-length λ (microns).	Refractive index n.	Verdet's constant δ.
(a)	·4434	1·4056$_8$	·0221$_6$
(b)	·3712	1·4138$_8$	·0332$_5$
(c)	·3325	1·4210$_3$	·0432$_3$

From (a) and (b) $\lambda_1 = ·1094_0\ \mu$ and K $= 5·401_2 \times 10^{-3}$.

,,   (a) and (c) $\lambda_1 = ·1093_6\ \mu$ and K $= 5·401_8 \times 10^{-3}$.

,,   (b) and (c) $\lambda_1 = ·1093_0\ \mu$ and K $= 5·403_2 \times 10^{-3}$.

The mean values of λ_1 and K are ·1094 μ and $5·40_2 \times 10^{-3}$ respectively, and therefore the magneto-optical dispersion of isopropyl butyrate at 11·4° C. in the region ·4647 μ to ·3257 μ is given by the equation

$$n\delta = 5·40_2 \times 10^{-3}\ \frac{\lambda^2}{\{\lambda^2 - (·1094)^2\}^2},$$

A comparison between values of δ calculated from this equation and those found experimentally is given in Table IV. (c).

TABLE IV. (c).

Wave-length λ (microns).	Verdet's constant δ (observed).	Verdet's constant δ (calculated.
·4647	·0199$_4$	·0199$_7$
·4337	·0232$_6$	·0232$_9$
·4167	·0254$_8$	·0254$_8$
·4016	·0277$_0$	·0277$_2$
·3883	·0299$_8$	·0299$_5$
·3766	·0321$_4$	·0321$_5$
·3662	·0343$_6$	·0343$_3$
·3565	·0365$_8$	·0365$_8$
·3478	·0387$_9$	·0387$_9$
·3399	·0410$_0$	·0409$_9$
·3257	·0454$_4$	·0454$_8$

The value of Verdet's constant of isopropyl butyrate at 11·4° C. for sodium light was calculated from the magneto-optical dispersion equation and found to be ·0119$_4$.

Ethyl Isobutyrate.

The experimental results are given in Table V. (a) and Graph 2 (d).

TABLE V. (a).

Values of Verdet's Constant at various wave-lengths for Ethyl Isobutyrate at 11° C.

Wave-length in microns.	Verdet's constant in min./cm. gauss.	Wave-length in microns.	Verdet's constant in min./cm. gauss.
·4574	·0199$_9$	·3383	·0400$_1$
·4363	·0222$_2$	·3308	·0422$_3$
·4270	·0233$_4$	·3239	·0444$_5$
·4183	·0244$_5$	·3175	·0466$_8$
·4020	·0266$_7$	·3117	·0489$_0$
·3883	·0289$_0$	·3062	·0511$_3$
·3759	·0311$_1$	·3010	·0533$_5$
·3650	·0333$_4$	·2960	·0555$_8$
·3554	·0355$_6$	·2916	·0578$_1$
·3466	·0377$_8$		

The constants K and λ of the magneto-optical dispersion equation of ethyl isobutyrate were calculated from the data given in Table V. (b).

TABLE V. (*b*).

	Wave-length λ (microns).	Refractive index n.	Verdet's constant δ.
(*a*)	·4363	$1·4026_5$	$·0222_2$
(*b*)	·3554	$1·4129_5$	$·0355_6$
(*c*)	·3010	$1·4263_8$	$·0533_5$

From (*a*) and (*b*) $\lambda_1 = ·1080_4\,\mu$ and $K = 5·227_5 \times 10^{-3}$
  ,,   (*a*) and (*c*) $\lambda_1 = ·1083_4\,\mu$ and $K = 5·222_9 \times 10^{-3}$
  ,,   (*b*) and (*c*) $\lambda_1 = ·1085_8\,\mu$ and $K = 5·216_8 \times 10^{-3}$.

The mean values of λ_1 and K are $·1083\,\mu$ and $5·22_2 \times 10^{-3}$ respectively, and the magneto-optical dispersion of ethyl isobutyrate at 11° C. for the range of spectrum between $·4574\,\mu$ and $·2916\,\mu$ is given by the equation

$$n\delta = 5·22_2 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (·1083)^2\}^2}.$$

A comparison between the values of δ calculated from this equation and those found experimentally is given in Table V. (*c*).

TABLE V. (*c*).

Wave-length λ (microns).	Verdet's constant δ (observed).	Verdet's constant δ (calculated).
·4574	$·0199_9$	$·0200_0$
·4183	$·0244_5$	$·0244_1$
·3883	$·0289_0$	$·0289_2$
·3650	$·0333_4$	$·0333_9$
·3383	$·0400_1$	$·0399_9$
·3239	$·0444_5$	$·0444_4$
·3117	$·0489_0$	$·0488_5$
·2916	$·0578_1$	$·0578_0$

The value of δ for sodium light calculated from the magneto-optical dispersion equation is $·0115_6$. Perkin's[5] value for the specific rotation of ethyl isobutyrate at 18·8° C. for sodium light is ·8771, and after correction to 11° C. the value becomes ·8841. Assuming Verdet's constant of water for sodium light to be ·01310 at 11° C., the value of Verdet's constant for ethyl isobutyrate at 11° C., deduced from Perkin's result, is ·01158.

Tertiary-Butyl Alcohol.

The results are given in Table VI. (*a*) and plotted in Graph 2 (*e*).

TABLE VI. (*a*).

Values of Verdet's Constant at various wave-lengths for Tertiary-Butyl Alcohol at 27° C.

Wave-length in microns.	Verdet's constant in min./cm. gauss.	Wave-length in microns.	Verdet's constant in min./cm. gauss.
$\cdot4580$	$\cdot0221_8$	$\cdot3341$	$\cdot0466_0$
$\cdot4394$	$\cdot0244_0$	$\cdot3279$	$\cdot0488_2$
$\cdot4225$	$\cdot0266_2$	$\cdot3222$	$\cdot0510_4$
$\cdot4083$	$\cdot0288_4$	$\cdot3170$	$\cdot0532_6$
$\cdot3956$	$\cdot0310_7$	$\cdot3119$	$\cdot0554_8$
$\cdot3842$	$\cdot0332_8$	$\cdot3072$	$\cdot0577_0$
$\cdot3738$	$\cdot0355_0$	$\cdot3027$	$\cdot0599_3$
$\cdot3645$	$\cdot0377_2$	$\cdot2985$	$\cdot0621_5$
$\cdot3557$	$\cdot0399_5$	$\cdot2943$	$\cdot0645_8$
$\cdot3480$	$\cdot0421_7$	$\cdot2908$	$\cdot0666_0$
$\cdot3410$	$\cdot0443_9$		

The above results can be represented within experimental error by the formula

$$n\delta\lambda^2 = K\left(\frac{\lambda^2}{\lambda^2-\lambda_1{}^2}\right)^2,$$

and the constants K and λ_1 were calculated from the values of λ, n, and δ given in Table VI. (*b*).

TABLE VI. (*b*).

	Wave-length λ (microns).	Refractive index n.	Verdet's constant δ.
(*a*)	$\cdot4394$	$1\cdot3923_5$	$\cdot0244_0$
(*b*)	$\cdot3480$	$1\cdot4040_7$	$\cdot0421_7$
(*c*)	$\cdot2985$	$1\cdot4161_2$	$\cdot0621_5$

From (*a*) and (*b*) $\lambda_1 = \cdot1148_4\,\mu$ and $K = 5\cdot693_8 \times 10^{-3}$.

,, (*a*) and (*c*) $\lambda_1 = \cdot1147_7\,\mu$ and $K = 5\cdot694_9 \times 10^{-3}$.

,, (*b*) and (*c*) $\lambda_1 = \cdot1147_0\,\mu$ and $K = 5\cdot697_2 \times 10^{-3}$.

The mean values of λ_1 and K are $1148\,\mu$ and $5\cdot69_5 \times 10^{-3}$ respectively, and the magneto-optical dispersion of

tertiary-butyl alcohol at 27° C. over the range of spectrum between ·4580 μ and ·2908 μ is given by

$$n\delta = 5·69_5 \times 10^{-3} \frac{\lambda^2}{\{\lambda^2 - (·1148)^2\}^2} .$$

A comparison of the values of δ calculated from this equation and those found experimentally is given in Table VI. (c).

<p align="center">TABLE VI. (c).</p>

Wave-length λ (microns).	Verdet's constant δ (observed).	Verdet's constant δ (calculated).
·4580	·0221$_8$	·0222$_3$
·4083	·0288$_4$	·0288$_7$
·3842	·0332$_6$	·0332$_7$
·3645	·0377$_3$	·0377$_0$
·3410	·0443$_9$	·0443$_3$
·3279	·0488$_2$	·0488$_6$
·3170	·0532$_6$	·0532$_1$
·3072	·0577$_0$	·0576$_8$
·2908	·0666$_0$	·0666$_1$

Perkin's [6] value for the specific rotation of tertiary-butyl alcohol at 27° C. for sodium light is ·9758. Assuming the value of Verdet's constant for water (sodium light) to be ·01307 at 27° C., the value of Verdet's constant of tertiary-butyl alcohol, deduced from Perkin's result, is ·0127$_6$, which is in fairly good agreement with the value ·0128$_0$ calculated from the magneto-optical dispersion equation.

The Ordinary Dispersion Equations.

The experimental results have shown that the magneto-optical dispersion of each liquid investigated can be explained by the presence of a single absorption band of wave-length λ_1 situated in the Schumann-Lyman region of the spectrum.

The ordinary dispersion of each liquid can also be represented within experimental error by the equation

$$n^2 - 1 = b_0 + \frac{b_1}{\lambda^2 - \lambda_1^2} ,$$

where λ_1 has the same value as that deduced from the magneto-optical experiments. The values of the constants b_0 and b_1 can be calculated from two values of the refractive index corresponding to two known wavelengths.

Methyl Malonate.

The pairs of values of λ and n used in the calculation of the constants b_0 and b_1 are given in Table VII. (a).

<p align="center">TABLE VII. (a).</p>

λ in microns.	n.	b_0.	$b_1 \times 10^2$.
·5016	$1·4201_3$	$·9750_7$	·9985
·3094	$1·4472_4$		
·4713	$1·4221_6$	$·9750_3$	·9978
·3109	$1·4467_4$		
·4249	$1·4262_1$	$·9748_6$	·9973
·3194	$1·4442_3$		
Mean values of constants ..		$·9749_9$	$·997_9$

The ordinary dispersion of methyl malonate at 14·7° C. is therefore represented by the equation

$$n^2 = 1·9749_9 + \frac{·997_9 \times 10^{-2}}{\lambda^2 - (·1101)^2}.$$

A comparison of the values of n calculated from this equation and those obtained experimentally is given in Table VII. (b).

<p align="center">TABLE VII. (b).</p>

λ (microns).	n (observed)	n (calculated).
·4922	$1·4207_9$	$1·4206_9$
·4531	$1·4235_8$	$1·4236_0$
·4378	$1·4249_8$	$1·4249_8$
·4063	$1·4283_3$	$1·4283_7$
·3656	$1·4342_6$	$1·4342_6$
·3405	$1·4390_5$	$1·4391_3$
·3247	$1·4429_1$	$1·4428_9$
·3064	$1·4481_4$	$1·4481_3$
·2997	$1·4503_9$	$1·4503_8$

Isopropyl Propionate.

The mean values of b_0 and b_1 calculated from the experimental results are $\cdot9084_9$ and $\cdot919_3 \times 10^{-2}$ respectively, and the equation representing the ordinary dispersion of isopropyl propionate at $9\cdot5°$ C. is

$$n^2 = 1\cdot9084_9 + \frac{\cdot919_3 \times 10^{-2}}{\lambda^2 - (\cdot1082)^2}.$$

A comparison of the values of n calculated from this equation and those obtained experimentally is given in Table VIII.

TABLE VIII.

λ (microns).	n (observed).	n (calculated).
$\cdot5016$	$1\cdot3953_0$	$1\cdot3952_8$
$\cdot4713$	$1\cdot3971_4$	$1\cdot3972_0$
$\cdot4540$	$1\cdot3984_8$	$1\cdot3984_4$
$\cdot4260$	$1\cdot4009_6$	$1\cdot4009_4$
$\cdot3974$	$1\cdot4040_1$	$1\cdot4040_5$
$\cdot3700$	$1\cdot4078_1$	$1\cdot4078_1$
$\cdot3498$	$1\cdot4111_5$	$1\cdot4112_3$
$\cdot3274$	$1\cdot4159_8$	$1\cdot4159_0$
$\cdot3064$	$1\cdot4216_0$	$1\cdot4213_9$

Isopropyl Butyrate.

The mean values of b_0 and b_1 calculated from the experimental results are $\cdot9263_7$ and $\cdot917_9 \times 10^{-2}$ respectively, and the ordinary dispersion equation for isopropyl butyrate at $11\cdot4°$ C. is

$$n^2 = 1\cdot9263_7 + \frac{\cdot917_9 \times 10^{-2}}{\lambda^2 - (\cdot1094)^2}.$$

A comparison of the values of n calculated from this equation and those obtained experimentally is given in Table IX.

TABLE IX.

λ (microns).	n (observed).	n (calculated).
$\cdot6678$	$1\cdot3955_7$	$1\cdot3955_4$
$\cdot5016$	$1\cdot4017_5$	$1\cdot4016_7$
$\cdot4713$	$1\cdot4035_8$	$1\cdot4035_9$
$\cdot4275$	$1\cdot4072_2$	$1\cdot4071_7$
$\cdot3881$	$1\cdot4114_8$	$1\cdot4115_8$
$\cdot3319$	$1\cdot4211_2$	$1\cdot4212_1$
$\cdot3109$	$1\cdot4264_1$	$1\cdot4264_5$
$\cdot3036$	$1\cdot4286_8$	$1\cdot4285_7$
$\cdot2824$	$1\cdot4361_0$	$1\cdot4358_9$

Ethyl Isobutyrate.

The mean values of b_0 and b_1 calculated from the experimental determinations of n at different wave-lengths are $\cdot9153_2$ and $\cdot932_4 \times 10^{-2}$ respectively, and therefore the dispersion equation of ethyl isobutyrate at $11°$ C. is

$$n^2 = 1\cdot9153_2 + \frac{\cdot932_4 \times 10^{-2}}{\lambda^2 - (\cdot1083)^2}.$$

A comparison of the values of n calculated from this equation and those obtained experimentally is given in Table X.

TABLE X.

λ (microns).	n (observed).	n (calculated).
$\cdot6678$	$1\cdot3918_0$	$1\cdot3916_9$
$\cdot4922$	$1\cdot3985_8$	$1\cdot3984_9$
$\cdot4275$	$1\cdot4034_5$	$1\cdot4035_1$
$\cdot3881$	$1\cdot4080_0$	$1\cdot4080_0$
$\cdot3484$	$1\cdot4143_5$	$1\cdot4143_4$
$\cdot3142$	$1\cdot4221_5$	$1\cdot4221_5$
$\cdot3064$	$1\cdot4244_2$	$1\cdot4243_7$
$\cdot2824$	$1\cdot4327_2$	$1\cdot4326_2$

Tertiary-Butyl Alcohol.

The mean values of b_0 and b_1 calculated from the experimental results are $\cdot8894_9$ and $\cdot884_1 \times 10^{-2}$ respectively, and the equation representing the dispersion of tertiary-butyl alcohol at $27°$ C. is

$$n^2 = 1\cdot8894_9 + \frac{\cdot884_1 \times 10^{-2}}{\lambda^2 - (\cdot1148)^2}.$$

A comparison of the values calculated from this equation and those obtained experimentally is given in Table XI.

TABLE XI.

λ (microns).	n (observed).	n (calculated).
$\cdot6678$	$1\cdot3819_6$	$1\cdot3820_0$
$\cdot4767$	$1\cdot3894_5$	$1\cdot3895_3$
$\cdot4249$	$1\cdot3936_4$	$1\cdot3936_7$
$\cdot3825$	$1\cdot3986_0$	$1\cdot3985_3$
$\cdot3414$	$1\cdot4053_2$	$1\cdot4053_5$
$\cdot3247$	$1\cdot4089_5$	$1\cdot4090_2$
$\cdot3074$	$1\cdot4135_6$	$1\cdot4135_9$
$\cdot2825$	$1\cdot4219_9$	$1\cdot4220_3$

The Calculation of e/m.

The value of e/m is given by the equation

$$\frac{e}{m} = \frac{2KC^2}{b_1},$$

and can, therefore, be deduced from the results of the magneto-optical and ordinary dispersion experiments.

The following values were obtained for e/m :—

Methyl malonate   .... $\cdot 87_6 \times 10^7$ E.M.U.
Isopropyl propionate   .... $1\cdot 01_0 \times 10^7$,,
Isopropyl butyrate.......... $1\cdot 02_7 \times 10^7$,,
Ethyl isobutyrate   .... $\cdot 97_8 \times 10^7$,,
Tertiary-butyl alcohol   ...... $1\cdot 12_4 \times 10^7$ ..

Discussion of Results.

An examination of the experimental results shows that the magneto-optical dispersion of methyl malonate, isopropyl propionate, isopropyl butyrate, ethyl isobutyrate, and tertiary-butyl alcohol over the range of spectrum investigated can be represented by the equation

$$n\delta = K\frac{\lambda^2}{(\lambda^2 - \lambda_1{}^2)^2},$$

where n and δ are the values of the refractive index and Verdet's constant for a wave-length λ, and λ_1 is the wave-length of the absorption band, which is situated in the Schumann-Lyman region of the spectrum. The values of K and λ_1 vary from liquid to liquid.

Similarly the refractive indices of the five liquids can be represented within experimental error by the equation

$$n^2 - 1 = b_0 + \frac{b_1}{\lambda^2 - \lambda_1{}^2},$$

where λ_1 has the same value as that deduced from the magneto-optical experiments, and the constants b_0 and b_1 depend on the nature of the liquid.

It is of interest to compare the results of the present investigation with those previously obtained for other substances in this laboratory. These results are collected in Table XII.

An examination of the table shows that when compounds of similar structure are compared, the values of λ_1 and K increase progressively with increasing molecular weight. The addition of a CH_2 group to the molecule increases the value of λ_1 by approximately $\cdot0012\,\mu$. It is also seen that λ_1 has practically the same value for the three butyl-alcohols, which have the same empirical formula but very different structures. In addition, the values of λ_1 are approximately the same for normal and iso esters of the same empirical formula.

TABLE XII.

Liquid.	Formula.	Molecular weight.	λ_1.	K.
n-Butyl alcohol [7].	$C_4H_9.OH.$	74	$\cdot1149\,\mu$	$5\cdot51_6 \times 10^{-3}$
Isobutyl alcohol [7].	$(CH_3)_2.CH.CH_2.OH.$	74	$\cdot1150\,\mu$	$5\cdot69 \times 10^{-3}$
Tertiary-butyl alcohol.	$(CH_3)_3.C.OH.$	74	$\cdot1148\,\mu$	$5\cdot69_5 \times 10^{-3}$
Ethyl propionate [8].	$C_2H_5.COO.C_2H_5.$	102	$\cdot1075\,\mu$	$5\cdot06_8 \times 10^{-3}$
n-Propyl propionate [9].	$C_2H_5.COO.C_3H_7.$	116	$\cdot1088\,\mu$	$5\cdot20_9 \times 10^{-3}$
Isopropyl propionate.	$C_2H_5.COO.CH(CH_3)_2.$	116	$\cdot1082\,\mu$	$5\cdot32_0 \times 10^{-3}$
Ethyl butyrate [10].	$C_3H_7.COO.C_2H_5.$	116	$\cdot1081\,\mu$	$5\cdot25_9 \times 10^{-3}$
Ethyl isobutyrate.	$CH(CH_3)_3.COO.C_2H_5.$	116	$\cdot1083\,\mu$	$5\cdot22_2 \times 10^{-3}$
Isopropyl butyrate.	$C_3H_7.COO.CH(CH_3)_2.$	130	$\cdot1094\,\mu$	$5\cdot40_2 \times 10^{-3}$
Methyl malonate.	$CH_2(COO.CH_3)_2.$	132	$\cdot1101\,\mu$	$5\cdot01_0 \times 10^{-3}$
Ethyl malonate [11].	$CH_2(COO.C_2H_5)_2.$	160	$\cdot1124\,\mu$	$5\cdot25_0 \times 10^{-3}$

For the organic compounds examined in this work Larmor's theory gives a satisfactory explanation of the variation of magneto-optical rotation with wave-length in the visible and near ultra-violet regions of the spectrum, but leads to a decidedly low value of e/m.

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XXXIX. *On Products of Laguerre Polynomials.*
By W. T. HOWELL, *M.Sc.**

§ 1. SEVERAL properties of the Laguerre polynomials have recently been studied [1], [2]. In this paper we shall be concerned with deductions that can be made from the definition of these polynomials by their generating function. The method used is that of the operational calculus [3], in which $f(x)$ is said to be operationally represented by $\phi(p)$ whenever

$$\phi(p) = p \int_0^\infty e^{-px} f(x) \, dx \quad . \quad . \quad . \quad . \quad (1)$$

and the integral converges. The relation (1) between $f(x)$ and $\phi(p)$ is denoted by

$$\phi(p) \doteqdot f(x) \quad \text{or} \quad f(x) \doteqdot \phi(p), \quad . \quad . \quad . \quad (2)$$

and, if the integral in (1) is absolutely convergent when p is real and equal to c, then $\phi(p)/p$ is an analytic function of the complex variable p when $\mathrm{R}(p) > c$, and it may be shown that

$$f(x) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} e^{px} \frac{\phi(p)}{p} \, dp. \quad . \quad . \quad . \quad (3)$$

With the above definition the following theorems may be proved :

$$\phi\left(\frac{p}{s}\right) \doteqdot f(sx), \quad s \text{ real and} > 0, \quad (4)$$

$$\frac{p}{p+\lambda} \phi(p+\lambda) \doteqdot e^{-\lambda x} f(x). \quad . \quad . \quad . \quad (5)$$

Further, if

$$\phi_1(p) \doteqdot f_1(x) \quad \text{and} \quad \phi_2(p) \doteqdot f_2(x),$$

then

$$\frac{1}{p} \phi_1(p)\phi_2(p) \doteqdot \int_0^x f_1(x-\xi) f_2(\xi) \, d\xi \quad . \quad . \quad . \quad (6)$$

and

$$\int_0^\infty \phi_1(u) f_2(u) \frac{du}{u} = \int_0^\infty f_1(x) \phi_2(x) \frac{dx}{x}. \quad . \quad . \quad (7)$$

§2. We define the Laguerre polynomial $L_n^\alpha(x)$ as usual by

$$\frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} t^n L_n^\alpha(x), \qquad \begin{matrix} |t| < 1 \\ \alpha > -1 \end{matrix}. \quad (8)$$

Multiplying both sides of (8) by $x^\beta e^{-px}$ and integrating from 0 to ∞ we get

$$\frac{\Gamma(\beta+1)\left\{1 - \left(\frac{p-1}{p}\right)t\right\}^{-\beta-1}}{p^{\beta+1}(1-t)^{\alpha-\beta}} = \sum_{n=0}^{\infty} t^n \int_0^\infty e^{-px} x^\beta L_n^\alpha(x)\, dx,$$

$$. \quad . \quad . \quad (9)$$

so that

$$\sum_{r=0}^{n} \frac{\Gamma(\alpha-\beta+r)\Gamma(\beta+1+n-r)}{\Gamma(\alpha-\beta)\,r!\,(n-r)!} \frac{(p-1)^{n-r}}{p^{n-r+\beta}} \doteq x^\beta L_n^\alpha(x). \quad (10)$$

We see at once from (10) that when $\beta = \alpha$ [4] *

$$x^\alpha L_n^\alpha(x) \doteq \frac{\Gamma(n+\alpha+1)}{n!} \frac{(p-1)^n}{p^{n+\alpha}}, \qquad . \quad . \quad (11)$$

and interpreting the left-hand side of (10) by (11) we get

$$L_n^\alpha(x) = \sum_{r=0}^{n} \frac{\Gamma(\alpha-\beta+r)}{\Gamma(\alpha-\beta)\,r!} L_{n-r}^\beta(x). \quad . \quad . \quad (12)$$

The result (12) is obviously very important, and, as far as I am aware, has never been given explicitly before. We may note here a few consequences of the simple representation given by (11). We have

$$\sum_{n=0}^{\infty} \frac{(-)^n n!}{\Gamma(n+\alpha+1)} x^\alpha L_n^\alpha(x) \doteq \frac{1}{2} \frac{1}{p} \frac{1}{p^{\alpha-1}} \frac{p}{p-\frac{1}{2}}.$$

Noting that $\frac{x^s}{\Gamma(s+1)} \doteq \frac{1}{p^s}$, and using (6), we find that

$$\sum_{n=0}^{\infty} \frac{(-)^n n!}{\Gamma(n+\alpha+1)} x^\alpha L_n^\alpha(x) = \frac{1}{2\Gamma(\alpha)} \int_0^\xi e^{\frac{\xi}{2}} (x-\xi)^{\alpha-1}\, d\xi$$

$$= \frac{e^{\frac{x}{2}}}{2\Gamma(\alpha)} \int_0^x e^{-\frac{\xi}{2}} \xi^{\alpha-1}\, d\xi. \quad . \quad (13)$$

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* Other values of β are considered in a future paper.

Similarly, we have

$$e^{-x}x^{\alpha}\sum_{n=0}^{\infty}\frac{(-)^n(2n)!}{\Gamma(2n+\alpha+1)}\mathrm{L}_{2n}^{\alpha}(2x)\doteqdot\frac{1}{2p}\frac{p}{(p+1)^{\alpha}}\frac{p^2+p}{p^2+1}.$$

Thus by variously interpreting the right-hand side, we have

$$e^{-x}x^{\alpha}\sum_{n=0}^{\infty}\frac{(-)^n(2n)!}{\Gamma(2n+\alpha+1)}\mathrm{L}_{2n}^{\alpha}(x)$$

$$=\frac{1}{2\Gamma(\alpha-1)}\int_0^x e^{-(x-\xi)}(x-\xi)^{\alpha-2}\sin\xi\,d\xi$$

$$=\frac{e^{-x}}{2\Gamma(\alpha)}\int_0^x e^{\xi}(x-\xi)^{\alpha-1}(\sin\xi+\cos\xi)\,d\xi$$

$$=\frac{1}{2\Gamma(\alpha)}\int_0^x e^{-\xi}\xi^{\alpha-1}\{\sin(x-\xi)+\cos(x-\xi)\}d\xi.$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{. . . (14)}$$

Also $$x_{\alpha}\sum_{n=0}^{\infty}\frac{t^n\,\mathrm{L}_n^{\alpha}(x)}{\Gamma(n+\alpha+1)}\doteqdot\frac{e^{t\left(1-\frac{1}{p}\right)}}{p^{\alpha}},$$

and since

$$(xt)^{\frac{\alpha}{2}}\mathrm{J}_{\alpha}(2\sqrt{tx})\doteqdot\left(\frac{t}{p}\right)^{\alpha}e^{-\frac{t}{p}},$$

we have

$$\sum_{n=0}^{\infty}\frac{t^n\,\mathrm{L}_n^{\alpha}(x)}{\Gamma(n+\alpha+1)}=e^t(xt)^{-\frac{\alpha}{2}}\mathrm{J}_{\alpha}(2\sqrt{tx})\;^{(5)},\qquad\text{. (15)}$$

where t may have any value.

By application of (4) and (5) we have

$$\frac{n!}{\Gamma(n+\alpha+1)}e^{-x}x^{\alpha}\mathrm{L}_n^{\alpha}(2x)\doteqdot\frac{p(p-1)^n}{(p+1)^{n+\alpha+1}}.$$

Expanding the right-hand side, and interpreting the terms by the relation

$$e^{-x}x^{\alpha}\mathrm{L}_n^{\alpha}(x)\doteqdot\frac{\Gamma(n+\alpha+1)}{n!}\frac{p^{n+1}}{(p+1)^{n+\alpha+1}},\qquad\text{. (16)}$$

we have

$$\mathrm{L}_n^{\alpha}(2x)=\sum_{r=0}^n\frac{(-x)^r}{r!}\mathrm{L}_{n-r}^{\alpha+r}(x).\qquad\text{. . . (17)}$$

Consider the representation

$$\frac{n!}{\Gamma(n+\alpha+1)} e^{-\frac{x}{\lambda}} \left(\frac{x}{\lambda}\right)^{\alpha} L_n^{\alpha}\left(\frac{x}{\lambda}\right) \doteq \frac{(p\lambda)^{n+1}}{(p\lambda+1)^{n+\alpha+1}}, \quad \lambda > 0.$$

The right-hand side may be written as

$$\frac{(p\lambda)^{n+1}}{(p+1)^{n+\alpha+1}} \left\{ 1 + \frac{p(\lambda-1)}{p+1} \right\}^{-n-\alpha-1}.$$

On expanding this and interpreting the terms by (16), we get the relation

$$n! \, e^{x-\frac{x}{\lambda}} \lambda^{-n-\alpha-1} L_n^{\alpha}\left(\frac{x}{\lambda}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(n+r)!}{r!} (1-\lambda)^r L_{n+r}^{\alpha}(x), \quad 0 < \lambda < 2. \quad (18)$$

We note also, by application of (6) to (11), the result

$$\frac{(n+m)! \, \Gamma(n+\alpha+1)\Gamma(m+\beta+1)}{n! \, m! \, \Gamma(n+m+\alpha+\beta+2)} x^{\alpha+\beta+1} L_{n+m}^{\alpha+\beta+1}(x)$$

$$= \int_0^x \xi^{\alpha} L_n^{\alpha}(\xi)(x-\xi)^{\beta} L_m^{\beta}(x-\xi) \, d\xi, \quad (19)$$

so that if we take $m=n$; $\beta=\alpha$, and use Bailey's [6] expression for the product $L_n^x(x)L_n^{\alpha}(y)$, we get

$$\frac{(2n)! \, \Gamma(n+\alpha+1)}{n! \, \Gamma(2n+2\alpha+2)} L_{2n}^{2\alpha+1}(x)$$

$$= \sum_{r=0}^{n} \frac{\Gamma(\alpha+r+1)}{r! \, \Gamma(2\alpha+2r+2)} x^{2r} L_{n-r}^{\alpha+2r}(x). \quad (20)$$

As an example of the application of (7) take

$$\phi_1(p) = p^s, \qquad\qquad f_1(x) = \frac{x^{-s}}{\Gamma(1-s)},$$

$$\phi_2(p) = \frac{\Gamma(n+\alpha+1)}{n!} \frac{p^{n+1}}{(p+1)^{n+\alpha-1}}, \quad f_2(x) = e^{-x} x^{\alpha} L_n^{\alpha}(x),$$

so that

$$\int_0^{\infty} e^{-x} x^{\alpha+s-1} L_n^{\alpha}(x) \, dx = \frac{\Gamma(n+\alpha+1)}{n! \, \Gamma(1-s)} \int_0^{\infty} p^{n-s}(p+1)^{-n-\alpha-1} \, dp$$

$$= \frac{\Gamma(s+\alpha)\Gamma(n-s+1)}{n! \, \Gamma(1-s)}.$$

§ 3. We shall now proceed to obtain an operational representation for the product of two Laguerre polynomials. We note in passing the Hill-Hardy generating function for products with different arguments, but with the same n and α, which is not sufficiently general for our purpose. We remark that by (12) we can express any product of the form $L_n^\alpha(ax)L_r^\beta(bx)$ as a series of products of the form $L_n^\alpha(ax)L_m^\alpha(bx)$, so that we shall be losing no generality by working with $L_n^\alpha(ax)L_m^\alpha(bx)$.

Thus supposing a and b both real and positive, we have by definition

$$\frac{e^{-\frac{axt}{1-t}-\frac{bxs}{1-s}}}{[(1-t)(1-s)]^{\alpha+1}} = \sum_{n=0}^\infty \sum_{s=0}^\infty t^n s^m \, L_n^\alpha(ax) \, L_m^\alpha(bx),$$

$$|t|<1 \; ; \; |s|<1 \; ; \; a>0 \; ; \; b>0.$$

Multiplying both sides of this equation by $x^\lambda e^{px}$ and integrating from 0 to ∞ , we get

$$\frac{\Gamma(\lambda+1)[p-(p-a)t-(p-b)s+(p-a-b)st]^{-\lambda-1}}{[(1-t)(1-s)]^{\alpha-\lambda}}$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty t^n s^m \int_0^\infty e^{-px} x^\lambda \, L_n^\alpha(ax) \, L_m^\alpha(bx) \, dx. \quad (21)$$

When $\lambda = \alpha$ *, the coefficient of $t^n s^m$ in the expression on the left of (21) is easily obtained, and we find therefore

$$x^\alpha \, L_n^\alpha(ax) \, L_m^\alpha(bx)$$

$$\doteqdot \frac{\Gamma(n+m+\alpha+1)}{n! \, m!} \frac{(p-a)^n (p-b)^m}{p^{n+m+\alpha}} F \left\{ \begin{array}{c} -n, \, -m \; ; \\ -n-m-\alpha \; ; \; \dfrac{p(p-a-b)}{(p-a)(p-b)} \end{array} \right\}, \quad (22)$$

and note that when λ is a positive integer

$$x^{\alpha+\lambda} \, L_n^\alpha(ax) \, L_m^\alpha(bx)$$

$$\doteqdot \frac{\Gamma(n+m+\alpha+1)}{n! \, m!} p \left(-\frac{d}{dp} \right)^\lambda \left[\frac{(p-a)^n (p-b)^m}{p^{n+m+\alpha}} \right.$$

$$\left. \times F \left\{ -n, \, -m \; ; \; -n-m-\alpha \; ; \; \frac{p(p-a-b)}{(p-a)(p-b)} \right\} \right]. \quad (23)$$

* Other values of λ than $\lambda = \alpha$ are to be considered in a future paper.

In (22) m and n are restricted to be positive integers, and by definition $\alpha > -1$.

We shall now make use of the operational representation (22) to derive expansions of products of Laguerre polynomials as sums of such polynomials. If we transform the argument in the hypergeometric function in (22) by the relation

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; -\frac{z}{1-z}\right) \quad (24)$$

and expand, then interpret the terms with the aid of the relation

$$\frac{n!}{\Gamma(n+\alpha+1)} x^\alpha L_n^\alpha(bx) \doteq \frac{(p-b)^n}{p^{n+\alpha}},$$

we see at once

$$\frac{m!}{\Gamma(n+\alpha+1)} L_n^\alpha(ax) L_m^\alpha(bx)$$

$$= \sum_{l=0}^{n} \sum_{r=0}^{l} (-)^r \binom{l}{r} \frac{(m+l-n-r)!\,[(ax)(bx)]^{n-l}}{l!\,(n-l)!\,\Gamma(n+\alpha+1-l)}$$

$$\times (ax)^r L_{m-n+l-r}^{\alpha+r+2n-2l}(bx)$$

or $\quad L_n^\alpha(x) L_m^\alpha(y)$

$$= \frac{\Gamma(n+\alpha+1)}{m!} \sum_{l=0}^{n} \sum_{r=0}^{l} (-)^r \binom{l}{r} \frac{(m-n+l-r)!\,(xy)^{n-l}}{l!\,(n-l)!\,\Gamma(n+\alpha+1-l)}$$

$$\times x^r L_{m-n+l-r}^{\alpha+r+2n-2l}(y). \quad (25)$$

Again from (5), (22), and (24) we have

$$e^{-(a+1)x} x^\alpha L_n^\alpha(ax) L_m^\alpha(x)$$

$$\doteq \frac{\Gamma(n+m+\alpha+1)}{n!\,m!} \frac{\Gamma(-n-m-\alpha)}{\Gamma(-n-\alpha)}$$

$$\times \sum_{l=0}^{n} (-)^l (-n)_l \frac{\Gamma(l-n-\alpha)}{l!\,\Gamma(l-n-m-\alpha)} \frac{p^{l+1}(p+a)^{m-n}}{(p+a+1)^{n+m+\alpha+1-l}}.$$

If we take $m = n$ and interpret the right-hand side by using

$$e^{-ax} x^\alpha L_n^\alpha(ax) \doteq \frac{\Gamma(n+\alpha+1)}{n!} \frac{p^{n+1}}{(p+a)^{n+\alpha+1}}, \quad . \quad (26)$$

and write y for ax, and r for $n-l$, we get

$$\mathrm{L}_n^\alpha(x)\,\mathrm{L}_n^\alpha(y) = \frac{\Gamma(n+\alpha+1)}{n!}\sum_{r=0}^n \frac{(xy)^r}{r!\,\Gamma(\alpha+r+1)}\,\mathrm{L}_{n-r}^{\alpha+2r}(x+y),$$
$$\cdots \quad (27)$$

a result previously given by Bailey [7].

Applying the transformation

$$\mathrm{F}(\alpha,\,\beta;\,\gamma;\,z) = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,\mathrm{F}(\alpha,\,\beta;\,1+\alpha+\beta-\gamma;\,1-z)$$
$$+ \frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\,(1-z)^{\gamma-\alpha-\beta}$$
$$\times \mathrm{F}(\gamma-\alpha,\,\gamma-\beta;\,1+\gamma-\beta-\alpha;\,1-z)$$

to the relation

$$e^{-x}x^\alpha\,\mathrm{L}_n^\alpha(x)\,\mathrm{L}_m^\alpha(x) \doteq \frac{\Gamma(n+m+\alpha+1)}{n!\,m!}\,\frac{p^{n+m+1}}{(p+1)^{n+m+\alpha+1}}$$
$$\times \mathrm{F}\left(-n,\,-m;\,-n-m-\alpha;\,1-\frac{1}{p^2}\right),\quad (28)$$

the right-hand side becomes

$$\frac{\Gamma(n+m+\alpha+1)}{n!\,m!}\left[\frac{\Gamma(-\alpha)\Gamma(-n-m-\alpha)\Gamma(\alpha+1)}{\Gamma(-m-\alpha)\Gamma(-n-\alpha)}\right.$$
$$\times \sum_{l=0}^n (-n)_l(-m)_l\,\frac{1}{\Gamma(\alpha+1+l)}\,\frac{p^{n+m+1-2l}}{(p+1)^{n+m+\alpha+1}}$$
$$+ \frac{\Gamma(\alpha)\Gamma(-n-m-\alpha)\Gamma(1-\alpha)}{\Gamma(-n)\Gamma(-m)\Gamma(-m-\alpha)\Gamma(-n-\alpha)}$$
$$\left.\times \sum_{l=0}^\infty \frac{\Gamma(l-m-\alpha)\Gamma(l-n-\alpha)}{l!\,\Gamma(1-\alpha+l)}\,\frac{p^{n+m+2\alpha+1-2l}}{(p+1)^{n+m+\alpha+1}}\right].$$

Using the properties of the Gamma function, and the representation (26), we find that the second summation vanishes since m and n are positive integers, and

$$\mathrm{L}_n^\alpha(x)\,\mathrm{L}_m^\alpha(y) = \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{\Gamma(n+m+\alpha+1)}$$
$$\times \sum_{l=0}^{n\ \mathrm{or}\ m} \frac{(n+m-2l)!\,x^{2l}}{l!\,(n-l)!\,(m-l)!\,\Gamma(\alpha+1+l)}\,\mathrm{L}_{n+m-2l}^{\alpha+2l}(x).\quad (29)$$

§4. We shall now give an alternative derivation of the operational representation of the product $e^{-x}x^\alpha\mathrm{L}_n^\alpha(x)\mathrm{L}_m^\alpha(x)$;

for this purpose we require the following integral due to Sonine [8]

$$\int_0^\infty e^{kx} e^{-x} x^\alpha \, \mathrm{L}_n^\alpha(x) \, dx = \frac{\Gamma(n+\alpha+1)}{n!} \frac{(-k)^n}{(1-k)^{n+\alpha+1}} ; \quad \mathrm{R}(k) < 1.$$

$$. \quad . \quad . \quad (30)$$

Multiplying both sides of the expression

$$n! \, m! \, e^{-x} x^\alpha \mathrm{L}_n^\alpha(x) \, \mathrm{L}_m^\alpha(x)$$

$$= \sum_{l=0}^\infty \frac{\Gamma(n+\alpha+l+1)\Gamma(m+\alpha+l+1)\Gamma(n+m+\alpha+2)}{l! \, \Gamma(\alpha+1+l)\Gamma(n+m+2\alpha+2+2l)}$$

$$. x^{\alpha+2l} \, \mathrm{L}_{n+m+\alpha+1}^{\alpha+2l}(x) \text{ [9]}$$

by e^{-px}, and integrating from 0 to ∞, we find in virtue of (30)

$$n! \, m! \int_0^\infty e^{-px} e^{-x} x^\alpha \mathrm{L}_n^\alpha(x) \mathrm{L}_m^\alpha(x) \, dx$$

$$= \sum_{l=0}^\infty \frac{\Gamma(n+\alpha+1+l)\Gamma(m+\alpha+1+l)}{l! \, \Gamma(\alpha+1+l)} \frac{(p-1)^{n+m+\alpha+1}}{p^{n+m+2\alpha+2+2l}}$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \frac{(p-1)^{n+m+\alpha+1}}{p^{n+m+2\alpha+2}}$$

$$\times \mathrm{F}\left(n+\alpha+1, \, m+\alpha+1 ; \, \alpha+1 ; \, \frac{1}{p^2}\right),$$

so that

$$e^{-x} x^\alpha \mathrm{L}_n^\alpha(x) \, \mathrm{L}_m^\alpha(x)$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{n! \, m! \, \Gamma(\alpha+1)} \frac{(p-1)^{n+m+\alpha+1}}{p^{n+m+2\alpha+1}}$$

$$\times \mathrm{F}\left(n+\alpha+1, \, m+\alpha+1 ; \, \alpha+1 ; \, \frac{1}{p^2}\right), \quad (31)$$

and since

$$\mathrm{F}(\alpha, \, \beta ; \, \gamma ; \, z) = (1-z)^{\gamma-\alpha-\beta} \mathrm{F}(\gamma-\alpha, \, \gamma-\beta ; \, \gamma ; \, z),$$

we have

$$e^{-x} x^\alpha \mathrm{L}_n^\alpha(x) \mathrm{L}_m^\alpha(x)$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{n! \, m! \, \Gamma(\alpha+1)} \frac{p^{n+m+1}}{(p+1)^{n+m+\alpha+1}}$$

$$\times \mathrm{F}\left(-n, \, -m ; \, \alpha+1 ; \, \frac{1}{p^2}\right). \quad (32)$$

We can now use some of the relations between the contiguous hypergeometric functions to deduce some of the properties of the above product.

Using the relation

$$\frac{\alpha z}{\gamma} F(\alpha+1, \beta+1; \gamma+1; z) = F(\alpha, \beta+1; \gamma; z) - F(\alpha, \beta; \gamma; z),$$

we may write the above representation in the form

$$-\frac{p^{n+m+3}}{(p+1)^{n+m+\alpha+1}} \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{(n+1)!\ m!\ \Gamma(\alpha+1)}$$
$$\times \left\{ F\left(-n-1, -m; \alpha; \frac{1}{p^2}\right) \right.$$
$$\left. - F\left(-n-1, -m-1; \alpha; \frac{1}{p^2}\right) \right\}$$
$$= -(m+\alpha)p \cdot \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha)}{(n+1)!\ m!\ \Gamma(\alpha)}\ \frac{p^{n+m+2}}{(p+1)^{n+m+\alpha+1}}$$
$$\times F\left(-n-1, -m; \alpha; \frac{1}{p^2}\right)$$
$$+(m+1)(p+1) \cdot \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{(n+1)!\ m+1)!\ \Gamma(\alpha)}$$
$$\times \frac{p^{n+m+3}}{(p+1)^{n+m+\alpha+2}} F\left(-n-1, -m-1; \alpha; \frac{1}{p^2}\right).$$

Whence interpreting by the aid of the rule $p\phi(p) \doteqdot \dfrac{d}{dx}(fx)$, we have

$$e^{-x}x^\alpha L_n^\alpha(x) L_m^\alpha(x) = -(m+\alpha) \frac{d}{dx} \{e^{-x}x^{\alpha-1} L_{n+1}^{\alpha-1}(x) L_m^{\alpha-1}(x)\}$$

$$+(m+1) \frac{d}{dx} \{e^{-x}x^{\alpha-1} L_{n+1}^{\alpha-1}(x) L_{m+1}^{\alpha-1}(x)\}$$

$$+e^{-x}x^{\alpha-1} L_{n+1}^{\alpha-1}(x) L_{m+1}^{\alpha-1}(x), \quad . \quad . \quad . \quad (33)$$

since $e^{-x}x^\alpha L_n^\alpha(x) L_m^\alpha(x) \to 0$ with x.

§5. Whether the operational method has any real advantage in this sort of work is a moot point. It may be remarked that equation (12) follows from the identity

$$\frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\alpha+1}} = \frac{1}{(1-t)^{\alpha-\beta}} \times \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\beta+1}}$$

by equating coefficients.

Equation (17) is easily derived from the identity

$$\frac{e^{-\frac{2xt}{1-t}}}{(1-t)^{\sigma+1}} = e^{-xt} \sum_{r=0}^{\infty} \frac{(-x)^r t^r}{(1-t)^{\alpha+r+1}},$$

$$\therefore \qquad \sum_{n=0}^{\infty} t^n L_n^{\alpha}(2x) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-x)^r t^r}{r!} t^s L_s^{\alpha+r}(x),$$

and equating coefficients of t^n, while (20) could probably be found similarly.

Apart from (15) and (27) the results are new. The operational representations (22), (28), and (30) when used in (3) afford new integral representations of the products of the generalized Laguerre polynomials, and it is hoped to make these the subject of a further paper.

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XL. *The Raman Spectra of some Chlorocarbonates.* *By* M. S. JOGLEKAR *.

THIS paper gives a preliminary study of the Raman Spectra of a few organic compounds represented by the structural formula $\begin{smallmatrix} O \diagdown \\ C-O-R. \\ Cl \diagup \end{smallmatrix}$ The liquids studied are methyl, ethyl, trichlor-methyl, and propyl, chlorocarbonates which were obtained from Frenkel and

Landau. The experimental technique used in obtaining the spectra of scattered light is too well known to be detailed here. In the present work a Hilger E_2 quartz spectrograph was used and two spectrograms were taken in each case, one with a filter consisting of an aqueous solution of sodium nitrite and another without any filter allowing the whole light from a mercury arc to illuminate the liquid. Calculations were made from the unmodified mercury lines as standards. Results recorded in Tables I. to IV. are correct to ± 5 cm.$^{-1}$.

In Tables I. and III. the mean values of the shifts observed with different exciting lines together with the visual estimate of their intensities are given. Tables II. and IV. are self explanatory.

TABLE I.

Methyl chlorocarbonate $\quad \overset{O}{\underset{Cl}{\diagdown}} C-O-CH_3.$
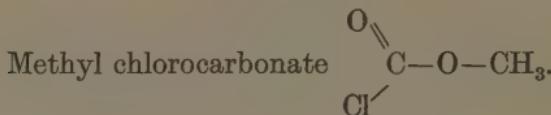
$\Delta\nu$ cm.$^{-1}=2962,$	1783,	1445,	1150,	944,	816,	476,
(2)	(1)	(1)	(1)	(2)	(4)	(8)
403,	271,	163.				
(3)	(1)	(1)				

The results show the usual characteristics of Raman Spectra of organic compounds belonging to a " series." The development of the C—H band [*] and the appearance [†] of new lines in the region $\Delta\nu=800$ to 1100 are due to the increasing number of carbon atoms in the radical " R." The presence of a heavy chlorine atom in the molecule gives rise to a number of low frequency shifts, which appear in all the spectra reported here.

Apart from these frequencies, the group $\overset{/\!O}{\underset{\diagdown Cl}{C}}$ gives rise to two shifts, whose values are $\Delta\nu=1783$ and 476 in methyl chlorocarbonate. The former is characteristic of

[*] A. S. Ganesan and V. N. Thatte, Phil. Mag, ser. 7, xiv. p. 1070 (1932).
[†] S. Venkateswaran and S. Bhagavantam, Ind. J. Phys. v. p. 129 (1930).

TABLE II.

Ethyl chlorocarbonate $\begin{array}{c} O \diagdown \\ C-O-C_2H_5. \\ Cl \diagup \end{array}$
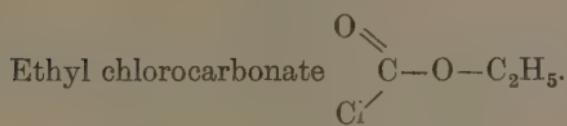
| Modified lines ν cm.$^{-1}$.	Exciting line λ.	$\Delta\nu$ cm.$^{-1}$.	Intensity.
21163	4358·3	1775	1
21484	,,	1454	$1\frac{1}{2}$
21722	4046·6	2983	5
21767	,,	2938	5
21833	4358·3	1105	$1\frac{1}{2}$
21929	,,	1009	2
22091	,,	847	4
22155	,,	783	0
22462	,,	476	8
22584	,,	354	5
23249	4046·6	1456	2
23290	4358·3	— 352	1
23420	,,	— 482	2
23604	4046·6	1101	2
23691	,,	1014	2
23794	4358·3	— 856	0
23856	4046·6	849	5
23925	,,	780	0
24037	4077·8	473	1
24142	4046·6	563 ?	1
24228	,,	477	7
24355	,,	350	3
24413	$\left\{\begin{array}{l} 3650 \cdot 1 \longrightarrow \\ 3654 \cdot 8 \longrightarrow \end{array}\right.$	$\left.\begin{array}{l} 2976 \\ 2941 \end{array}\right\}$	4 broad
24452	3650·1	2937	2

$\Delta\nu$ cm.$^{-1}$ = 2979, 2939, 1775, 1455, 1103, 1011, 850, 781,
      (5)   (5)   (1)   (2)   (2)   (2) (5) (0)

   563 ?, 477, 352.
    (1)   (7)   (3)

TABLE III.

Trichlor-methyl chlorocarbonate
$$\underset{Cl}{\overset{O}{\underset{\diagdown}{\overset{\diagup}{C}}}}\!\!\!-\!\!O\!-\!CCl_3.$$
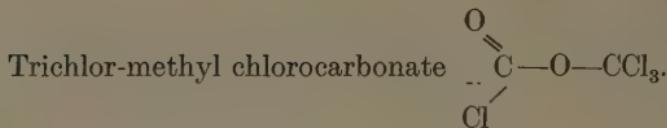
$\Delta\nu$ cm.$^{-1}$=	1799,	906,	799,	729,	493,	398,
	(0)	(2)	(1)	(1)	(6)	(1)
	347,	231,	172.			
	(6)	(3)	(1)			

TABLE IV.

Propyl chlorocarbonate
$$\underset{Cl}{\overset{O}{\underset{\diagdown}{\overset{\diagup}{C}}}}\!\!\!-\!\!O\!-\!C_3H_7.$$
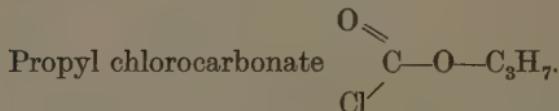
Modified lines ν cm.$^{-1}$.	Exciting line λ.	$\Delta\nu$ cm.$^{-1}$.	Intensity.
20000	4358·3	2938	1
21160	,,	1778	1
21480	,,	1458	2
21726	4046·6	2979	3
21768	,,	2937	5
21816	,,	2889	3
21887	4358·3	1051	2
22036	,,	902	3
22125	,,	813	2
22463	,,	475	6
22551	,,	387	2
22645	,,	293	2
23251	4046·6	1454	$1\frac{1}{2}$
23804	,,	902	2
23897	,,	808	$1\frac{1}{2}$
24234	,,	471	4

$\Delta\nu$ cm.$^{-1}$= 2799, 2937, 2889, 1778, 1456, 1051, 902,
　　　　　(3)　　(5)　　(3)　　(1)　　(2)　　(2)　　(3)

　　　　810, 473, 387, 293.
　　　　(2)　(5)　(2)　(2)

$C=O$ while the latter can be identified with the $C—Cl$ linkage. Ethyl and propyl chlorocarbonates give these shifts just of the same magnitude, but in the case of trichlor-methyl they have slightly higher values, viz., $\Delta\nu=1799$ and 493. The constancy of these two shifts seems to be a peculiarity of this class of compounds. It is probably due to the fact that the group $C\overset{\displaystyle /\!\!O}{\underset{\displaystyle \diagdown Cl}{}}$ is always linked to an atom of oxygen in a chlorocarbonate molecule. The frequency * of about $\Delta\nu=600$ which is associated with the configuration $-\overset{\displaystyle O}{\overset{\|}{C}}-$ is, however, absent here.
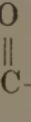
In a recent communication to this Journal the author has reported the Raman Spectra of a number of acid chlorides belonging to the aliphatic and aromatic classes. Acid chlorides also contain the group $C\overset{\displaystyle /\!\!O}{\underset{\displaystyle \diagdown Cl}{}}$ which is present in chlorocarbonates. The characteristic frequencies of this group in acid chlorides vary within certain limits, however, their mean values in aliphatic compounds are $\Delta\nu=1790$ and 425 cm.$^{-1}$. The point of interest is that whereas the frequency of the carbonyl group in $C\overset{\displaystyle /\!\!O}{\underset{\displaystyle \diagdown Cl}{}}$ is the same both for acid-chlorides and chlorocarbonates, that for the $C—Cl$ linkage is considerably different in these two classes of compounds.
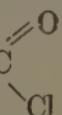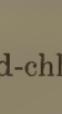
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* S. Venkateswaran and S. Bhagavantam, Ind. J. Phys. v. p. 129 (1930).

XLI. *On some Two-Dimensional Problems in Heat Conduction. By* Arnold N. Lowan †.

THE object of the present paper is the extension of the method presented in a number of earlier papers ‡ dealing with the uni-dimensional flow of heat, to the case of two-dimensional problems. The reader is referred to the aforementioned papers for the derivation of some of the results to be subsequently used.

Consider the problem of heat conduction in an infinite parallelepiped bounded by the planes $x=0$, $x=a$, $y=0$, $y=b$. For the sake of generality, let us assume that the solid contains a continuous distribution of heat generating sources, characterized by the function $\psi(x, y, t)$, and that the initial temperature distribution is $f(x, y)$. Concerning the boundary conditions, we assume that the faces $y=0$ and $y=b$ are impervious to heat ; that for the faces $x=0$ and $x=a$ the expressions $\alpha_{1, 2}\partial T/\partial x+\beta_{1, 2}T$ are prescribed functions $\phi_{1, 2}(y, t)$. The temperature $T(x, y, t)$ must then satisfy the following system :—

$$A \begin{cases} \kappa(\partial^2/\partial x^2+\partial^2/\partial y^2)T(x, y, t)=(\partial/\partial t)T(x, y, t)-\psi(x, y, t), \\ \qquad\qquad 0<x<a ; \quad o<y<b ; \quad t>o, \quad . \quad . \quad . \quad (1) \\ \text{Lim } T(x, y, t)=f(x, y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \\ t\to 0 \\ (\partial/\partial y)T(x, y, t)=0 \text{ for } y=o \text{ and } y=b, \quad . \quad . \quad . \quad (3) \\ (\alpha_1\partial/\partial x+\beta_1)T(x, y, t)=\phi_1(y, t) \text{ for } x=0, \quad . \quad . \quad . \quad (4) \\ (\alpha_2\partial/\partial x+\beta_2)T(x, y, t)=\phi_2(y, t) \text{ for } x=a. \quad . \quad . \quad . \quad (5) \end{cases}$$

It may be remarked that the boundary conditions (4) and (5) contain as special cases the customary conditions encountered in practice. Thus $\alpha=0$ amounts to prescribing the surface temperature ; $\beta=0$ amounts to prescribing the superficial temperature gradient ; if neither α nor β is $=0$ the condition is that of radiation into a medium of prescribed temperature ; finally $\beta=0$ and $\phi(y, t)=0$ signifies that the boundary is impervious to heat.

To solve the system (1)–(5) we put :

$$T(x, y, t)=u(x, y, t)+v(x, y, t), \quad . \quad . \quad . \quad (6)$$

where

$$\kappa(\partial^2/\partial x^2 + \partial^2/\partial y^2)u(x, y, t) = (\partial/\partial t)u(x, y, t), \quad \ldots \quad (7)$$
$$0 < x < a; \quad 0 < y < b; \quad t > 0,$$

B

$$\operatorname*{Lim}_{t \to 0} u(x, y, t) = 0, \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$(\partial/\partial y)u(x, y, t) = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b, \quad \ldots \quad (9)$$

$$(\alpha_1 \partial/\partial x + \beta_1)u(x, y, t) = \phi_1(y, t) \quad \text{for} \quad x = 0, \quad \ldots \quad (10)$$

$$(\alpha_2 \partial/\partial x + \beta_2)u(x, y, t) = \phi_2(y, t) \quad \text{for} \quad x = a, \quad \ldots \quad (11)$$

and

$$\kappa(\partial^2/\partial x^2 + \partial^2/\partial y^2)v(x, y, t) = (\partial/\partial t)v(x, y, t) - \psi(x, y, t),$$
$$\ldots, \quad (12)$$
$$0 < x < a; \quad 0 < y < b; \quad t > 0,$$

C

$$\operatorname*{Lim}_{t \to 0} v(x, y, t) = f(x, y), \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$(\partial/\partial y)v(x, y, t) = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = b, \quad \ldots \quad (14)$$

$$(\alpha_1 \partial/\partial x + \beta_1)v(x, y, t) = 0 \quad \text{for} \quad x = 0, \quad \ldots \quad \ldots \quad (15)$$

$$(\alpha_2 \partial/\partial x + \beta_2)v(x, y, t) = 0 \quad \text{for} \quad x = a. \quad \ldots \quad \ldots \quad (16)$$

We proceed to solve the system B. Let

$$\mathrm{L}\{u(x, y, t\} = \int_0^\infty e^{-pt}u(x, y, t)\, dt = u^*(x, y, p), \quad (17)$$

$$\mathrm{L}\{\phi_1(y, t)\} = \phi_1^*(y, p), \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$\mathrm{L}\{\phi_2(y, t)\} = \phi_2^*(y, p). \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

If we then operate on (7), (9), (10), and (11) with the Laplace operator L thus defined, and make use of the identity

$$\mathrm{L}\{(\partial/\partial t)u(x, y, t)\} = pu^*(x, y, p) - u(x, y, o),$$
$$= pu^*(x, y, p) \quad \ldots \quad \ldots \quad (20)$$

(where we have made use of (8)).

The function $u^*(x, y, p)$ must satisfy the system

D

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + q^2\right)u^*(x, y, p) = 0, \quad \ldots \quad \ldots \quad (21)$$

$$\frac{d}{dy}u^*(x, y, p) = 0 \quad \text{for} \quad y = o \quad \text{and} \quad y = b, \quad \ldots \quad (22)$$

$$\left(\alpha_1 \frac{\partial}{\partial x} + \beta_1\right)u^*(x, y, p) = \phi_1^*(y, p) \quad \text{for} \quad x = 0, \quad (23)$$

$$\left(\alpha_2 \frac{\partial}{\partial x} + \beta_2\right)u^*(x, y, p) = \phi_2^*(y, p) \quad \text{for} \quad x = a, \quad (24)$$

2 E 2

where we have put

$$p = -\kappa q^2. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The expression

$$u^*(x,\, y,\, p) = \sum_{n=0}^{\infty} (A_n \sin \lambda_n x + B_n \cos \lambda_n x) \cos \mu_n y \quad (26)$$

satisfies (21) and (22) provided

$$\mu_n = \frac{n\pi}{b}, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and

$$\gamma_n^2 + \mu_n^2 = q^2. \quad . \quad . \quad . \quad . \quad (28)$$

In order to satisfy (23) and (24) we must have

$$\sum_{n=0}^{\infty} (\alpha_1 A_n \lambda_n + \beta_1 B_n) \cos \mu_n y = \phi_1^*(y,\, p) \quad . \quad . \quad (29)$$

and

$$\sum_{n=0}^{\infty} \{(\beta_2 A_n - \alpha_2 B_n \lambda_n) \sin \lambda_n a + (\alpha_2 A_n \lambda_n + \beta_2 B_n) \cos \lambda_n a\} \cos \mu_n y$$
$$= \phi_2^*(y,\, p). \quad . \quad . \quad (30)$$

Let

$$\phi_{1,\,2}^*(y,\, p) = \frac{1}{b} \sum_{n=0}^{\infty} \delta_n \cos \mu_n y \int_0^b \phi_{1,\,2}^*(\eta,\, p) \cos \mu_n \eta \, d\eta$$
$$= \sum_{n=0}^{\infty} C_n^{1,\,2}(p) \cos \mu_n y, \quad (\text{say}) \quad . \quad (31)$$

where $\delta_0 = 1$ and $\delta_n = 2$ for $n \neq 0$.

It then follows from (29) and (30)

$$\alpha_1 \lambda_n A_n + \beta_1 B_n = C_n^1(p)$$
$$(\alpha_2 \lambda_n \cos \lambda_n a + \beta_2 \sin \lambda_n a) A_n$$
$$\quad\quad + (\beta_2 \cos \lambda_n a - \alpha_2 \lambda_n \sin \lambda_n a) B_n = C_n^2(p), \quad . \quad . \quad (32)$$

whence

$$A_n = \frac{(\beta_2 \cos \lambda\, a - \alpha_2 \lambda_n \sin \lambda_n a)\delta_n}{bZ(\lambda_n)} \int_0^b \phi_1^*(\eta,\, p) \cos \mu_n \eta \, d\eta$$
$$- \frac{\beta_1 \delta_n}{bZ(\lambda_n)} \int_0^b \phi_2^*(\eta,\, p) \cos \mu_n \eta \, d\eta. \quad (33)$$

$$B_n = \frac{\alpha_1 \lambda_n \delta_n}{b Z(\lambda_n)} \int_0^b \phi_2^*(\eta, p) \cos \mu_n \eta \, d\eta$$

$$- \frac{(\alpha_2 \lambda_n \cos \lambda_n a + \beta_2 \sin \lambda_n a) \delta_n}{b Z(\lambda_n)} \int_0^b \phi_1^*(\eta, p) \cos \mu_n \eta \, d\eta,$$

$$\text{. . . (34)}$$

where we have written

$$Z(\lambda_n) = \lambda_n(\alpha_1 \beta_2 - \alpha_2 \beta_1) \cos \lambda_n a - (\alpha_1 \alpha_2 \lambda_n^2 + \beta_1 \beta_2) \sin \lambda_n a.$$

$$\text{. . . (35)}$$

With the aid of (33) and (34) the final solution of the system D becomes :

$$u^*(x, y, p) = \sum_{n=0}^{\infty} \delta_n \cos \mu_n y$$

$$\times \frac{(\alpha_1 \lambda_n \cos \lambda_n x - \beta_1 \sin \lambda_n x)}{b Z(\lambda_n)} \int_0^b \phi_2^* \cos \mu_n \eta \, d\eta$$

$$+ \sum_{n=0}^{\infty} \frac{\delta_n}{b \overline{Z}(\lambda_n)} \{\beta_2 \sin \lambda_n(x-a) - \alpha_2 \lambda_n \cos \lambda_n(x-a)\}$$

$$\times \cos \mu_n y \int_0^b \phi_1^*(\eta, p) \cos \mu_n \eta \, d\eta. \quad (36)$$

Our problem now reduces to the inversion of the Laplace transform (17) where $u^*(x, y, p)$ is given by (36). This can be accomplished with the aid of Borel's Theorem. Since this important theorem will be frequently employed in the present paper, its formulation will be given here. If

$$L\{f_1(t)\} = f_1^*(p)$$

and

$$L\{f_2(t)\} = f_2^*(p),$$

then

$$L\left\{ \int_0^t f_1(\tau) f_2(t-\tau) \, d\tau \right\} = L\left\{ \int_0^t f_1(t-\tau) f_2(\tau) \, d\tau \right\}$$
$$= f_1^*(p) \cdot f_2^*(p).$$

Let us define the functions $U_n(x, t)$ by :

$$\int_0^{\infty} \exp(-pt) U_n(x, t) \, dt = \frac{\sin \lambda_n x}{Z(\lambda_n)} = U_n^*(x, p) \text{ (say). } \quad (37)$$

Then making use of Borel's Theorem (36) yields the complete solution of the system B in the compact form

$$u(x, y, t) = \sum_{n=0}^{\infty} \frac{\delta_n}{b} \cos \mu_n y \int_0^{\cdot} \cos \mu_n \eta \, d\eta$$

$$\times \int_0^t \{[\alpha_1 \partial/\partial x - \beta_1) U_n(x, t-\tau) \cdot \phi_2(\eta, \tau)]$$

$$+ [(-\alpha_2 \partial/\partial x + \beta_2) U_n(x-a, t-\tau) \cdot \phi_1(\eta, \tau)]\} d\tau. \quad (38)$$

Calculation of the $U_n(x, t)$ functions :—

Let us rewrite (37) in the form

$$p \int_0^\infty \exp(-pt) U_n(x, t) \, dt = \frac{p \sin \lambda_n x}{Z(\lambda_n)} = \frac{Y_n(p)}{Z_n(p)} \quad \text{(say)}.$$

$$\cdot \quad \cdot \quad \cdot \quad (37')$$

In the language of the Operational Calculus, the expression $\dfrac{Y_n(p)}{Z_n(p)}$ is the operational solution corresponding to the function $U_n(x, t)$. The inversion of (37) yields

$$U_n(x, t) = \frac{Y_n(0)}{Z_n(0)} + \sum \frac{Y_n(p_i)}{p_i Z'(p_i)} \exp(p_i t), \quad . \quad (39)$$

where the summation is extended over the roots of $Z(p) = 0$ or

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\lambda \cos a\lambda - (\alpha_1\alpha_2\lambda^2 + \beta_1\beta_2) \sin \lambda a = 0, \quad . \quad . \quad (40)$$

provided (40) has no repeated roots, a condition which can be easily verified.

We have $Y_n(0) = 0$. Further, from

$$\lambda^2 + \mu^2 = q^2 = -\frac{p}{\kappa_1}, \quad \text{we get} \quad \frac{d\lambda}{dp} = -\frac{1}{2\lambda\kappa},$$

and therefore, in view of (35),

$$\frac{\partial Z}{\partial p} = \frac{dZ}{d\lambda} \cdot \frac{d\lambda}{dp} = -\frac{1}{2\lambda\kappa} \frac{dZ}{d\lambda}$$

$$= \frac{-1}{2\lambda\kappa} \{[A - a(B\lambda^2 + C)] - (Aa + 2B)\} \sin a\lambda,$$

where we have put

$$A = \alpha_1\beta_2 - \alpha_2\beta_1; \quad B = \alpha_1\alpha_2; \quad C = \beta_1\beta_2.$$

Making use of (40), we get

$$\left(\frac{\partial Z}{\partial p}\right)_{p=p_i}$$

$$= \frac{-b}{2\lambda\kappa} \left\{ \frac{A - a(B\lambda^2 + C)}{A\lambda} (B\lambda^2 + C) - (Aa + 2B) \right\} \sin a\lambda_i,$$

whence finally

$$\frac{Y_n(p_i)}{p_i Z'(p_i)}$$
$$= \frac{-2\kappa A\lambda_i^2 \sin x\lambda_i}{\{(A-aC-aB\lambda_i^2)(B\lambda^2+C) - A\lambda_i(Aa+2B)\}\sin a\lambda_i}, \quad (41)$$

and therefore

$$U_n(x, t)$$
$$= \Sigma \frac{2A\kappa\lambda_i^2 \sin \lambda_i x . \exp\left[-\kappa(\mu_n^2 + \lambda_i^2)t\right]}{[A\lambda_i(Aa+2B) - (B\lambda_i^2+C)(A-aC-aB\lambda_i^2)]\sin a\lambda_i},$$
$$\qquad\qquad\qquad\qquad . \quad . \quad . \quad (42)$$

where the summation extends over the roots of (40).

In (38) and (42) we have the complete solution of the system B.

Case $b = \infty$.

Consider the identity

$$F(y) = \frac{1}{\pi} \int_0^\infty d\xi \int_{-\infty}^{+\infty} F(\eta) \cos \xi(y-\eta) \, d\eta. \quad . \quad (43)$$

When $F(-y) = F(y)$ this reduces to

$$F(y) = \frac{2}{\pi} \int_0^\infty d\xi \int_0^\infty F(\eta) \cos \xi y \cos \xi\eta \, d\eta. \quad . \quad (44)$$

The expression

$$\frac{2}{\pi} \int_0^\infty d\xi \int_0^\infty (A \sin \lambda x + B \cos \lambda x) \cos \xi y \cos \xi\eta \, d\eta, \quad (45)$$

where $\lambda^2 + \xi^2 = q^2$ is a solution of (21) satisfying the boundary condition (22).

In view of the boundary condition (22) it is clear that we may assume that the functions $\phi_1(y, t)$ and $\phi_2(y, t)$ are defined for $y < 0$ and that, furthermore, $\phi_{1, 2}(-y, t) = \phi_{1, 2}(y, t)$. This property is evidently shared by their Laplace transforms $\phi_{1, 2}^*(y, p)$. In view of (44) it follows that the expression (45) will satisfy the boundary conditions (23) and (24) provided the expression

$$A \sin \lambda x + B \cos \lambda x = \theta(x)$$

satisfies the conditions

$$\left.\begin{array}{ll} (\alpha_1 \partial/\partial x + \beta_1)\theta(x) = \phi_1^*(\eta, p) & \text{for } x = 0, \\ (\alpha_2 \partial/\partial x + \beta_2)\theta(x) = \phi_2^*(\eta, p) & \text{for } x = a. \end{array}\right\} . \quad (46)$$

Equations (46) are similar to (32) except that λ takes the place of λ_n and C_n^1, C_n^2 are replaced by $\phi_1^*(\eta, p)$ and $\phi_2^*(\eta, \pi)$. Thus

$$A = \frac{\beta_2 \cos \lambda a - \alpha_2 \lambda \sin \lambda a}{Z(\lambda)} \phi_1^*(\eta, p) - \frac{\beta_1}{Z(\lambda)} \phi_2^*(\eta, p), \quad . \quad (47)$$

$$B = -\frac{\alpha_2 \lambda \cos \lambda a + \beta_2 \sin \lambda a}{Z(\lambda)} \phi_1^*(\eta, p) + \frac{\alpha_1 \lambda}{Z(\lambda)} \phi_2^*(\eta, p). \quad (48)$$

With the aid of (47) and (48) we get

$$A \sin \lambda x + B \cos \lambda x = \frac{\alpha_1 \lambda \cos \lambda x - \beta_1 \sin \lambda x}{Z(\lambda)} \phi_1^*(\eta, p)$$

$$+ \frac{\beta_2 \sin \lambda(x-a) - \alpha_2 \cos \lambda(x-a)}{Z(\lambda)} \phi_2^*(\eta, p). \quad (49)$$

Substituting (49) in (45) the complete solution of the system D, for the case $b = \infty$ is obtained in the form

$$u_{b=\infty}^*(x, y, p)$$

$$= \frac{2\alpha_1}{\pi} \int_0^\infty d\xi \int_0^\infty \frac{\lambda \cos \lambda x}{Z(\lambda)} \phi_1^*(\eta, p) \cos \xi \eta \cos \xi y \, d\eta$$

$$- \frac{2\beta_1}{\pi} \int_0^\infty d\xi \int_0^\infty \frac{\sin \lambda x}{Z(\lambda)} \phi_1^*(\eta, p) \cos \xi \eta \cos \xi y \, d\eta$$

$$+ \frac{2\beta_2}{\pi} \int_0^\infty d\xi \int_0^\infty \frac{\sin \lambda(x-a)}{Z(\lambda)} \phi_2^*(\eta, p) \cos \xi y \cos \xi \eta \, d\eta$$

$$- \frac{2\alpha_2}{\pi} \int_0^\infty d\xi \int_0^\infty \frac{\lambda \cos \lambda(x-a)}{Z(\lambda)} \phi_2^*(\eta, p) \cos \xi \eta \cos \xi y \, d\eta.$$

$$. \quad . \quad . \quad (50)$$

As in the previous case, the problem now reduces to the inversion of the Laplace transform (17), where $U^*(x,y,p)$ is given by (50).

If, as before, we define the function $U(x, t)$ by the equation

$$\int_0^\infty \exp[-pt]U(x, t) \, dt = \frac{\sin \lambda x}{Z(\lambda)}, \quad . \quad . \quad (51)$$

the complete solution of the system B assumes the form

$$u_{b=\infty}^*(x, y, t)$$

$$= \frac{2}{\pi} \int_0^\infty d\xi \int_0^\infty d\eta \int_0^t \{(\alpha_1 \partial/\partial x - \beta_1)U(x, t-\tau)\phi_1(\eta, \tau)$$

$$+ (\beta_2 \partial/\partial x - \alpha_2)U(x-a, t-\tau)\phi_2(\eta, \tau) \cos \xi y \cos \xi \eta \, d\eta.$$

$$. \quad . \quad . \quad (52)$$

It is clear that the function $U(x, t)$ may be obtained from our previous $U_n(x, t)$ by replacing $\mu_n{}^2$ in the exponential factor by ξ^2.

Case $a = \infty$.

It is easily seen from (36) that both $u^*(x, y, p)$ and $(\partial/\partial x)u^*(x, y, p)$ and, therefore, also $u(x, y, t)$ and $(\partial/\partial x)u(x, y, t)$ tend to zero as $a \to \infty$, a fact which is, of course, obvious from physical considerations.

It therefore follows that in the case under consideration the boundary condition at $x = \infty$ may be replaced by

$$\underset{x \to \infty}{\text{Lim}} \; u(x, y, t) = 0. \quad \ldots \quad \ldots \quad (53)$$

The expression

$$u^*(x, y, p) = \sum_{n=0}^{\infty} A_n \exp\left[-\lambda_n x\right] \cos \mu_n y. \quad . \quad (54)$$

is a solution of (21) satisfying (22) and (53) provided

$$\mu_n = \frac{n\pi}{b} \quad \text{and} \quad p = \kappa q^2 = \kappa(\lambda_n^2 - \mu_n^2). \quad . \quad . \quad (55)$$

In order to satisfy (23) we must have

$$\sum_{n=0}^{\infty} A_n(\beta - \alpha\lambda_n) \cos \mu_n y = \phi^*(y, p)$$

$$= \sum_{n=0}^{\infty} \frac{\delta_n}{b} \cos \mu_n y \int_0^b \phi^*(\eta, p) \cos \mu_n \eta \, d\eta, \quad (56)$$

whence ultimately

$$\mu^*(x, y, p) = \sum_{n=0}^{\infty} \frac{\delta_n}{b} \frac{\exp\left[-\lambda_n x\right]}{h - \lambda_n} \cos \mu_n y$$

$$\times \int_0^b \frac{1}{\alpha} \phi^*(\eta, p) \cos \mu_n \eta \, d\eta, \quad (57)$$

where we have put $h = \dfrac{\beta}{\alpha}$.

If we now introduce the new function $W_n(x, t)$ defined by

$$\int_0^{\infty} \exp\left[-pt\right] W_n(x, t) \, dt = \frac{\exp\left[-\lambda_n x\right]}{h - \lambda_n}$$

$$= \frac{\exp\left[\dfrac{-x}{\sqrt{\kappa}} \sqrt{p + \kappa\mu_n^2}\right]}{h - \dfrac{1}{\sqrt{\kappa}} \sqrt{p + \kappa\mu_n^2}}, \quad . \quad (58)$$

then the application of Borel's Theorem to (57) yields

$$\underset{a \to \infty}{\text{Lim}}\, u(x, y, t) = \sum_{n=0}^{\infty} \frac{\delta_n}{b} \cos \mu_n y$$

$$\times \frac{1}{\alpha} \int_0^t \int_0^b W_n(x, t - \tau)\phi(\eta, \tau) \cos \mu_n \eta \, d\eta \, d\tau, \quad (59)$$

and the problem now reduces to the calculation of $W_n(x, t)$.

For this purpose consider the problem of heat conduction in a semi-infinite solid initially at $0°$, radiating into a medium at $1°$. The temperature $w(x, t)$ must then satisfy the system

$$\left.\begin{array}{c} (\partial/\partial t)w(x, t) - \kappa(\partial^2/\partial x^2)w(x, t) = 0, \\ x > 0, \quad t > 0, \\ \underset{t \to \infty}{\text{Lim}}\, w(x, t) = 0, \\ (-\partial/\partial x + \hbar)w(x, t) = \hbar \quad \text{for } x = 0. \end{array}\right\} \text{E}$$

The Laplace transform of $u(x, t)$ is readily obtained in the form.

$$w^*(x, p) = \frac{h}{p} \cdot \frac{\exp[-qx]}{h - q},$$

$$= \frac{h}{p} \cdot \frac{\exp\left[\dfrac{-x}{\sqrt{\kappa}} \sqrt{p}\right]}{h - \dfrac{1}{\sqrt{\kappa}} \sqrt{p}}, \quad . \quad . \quad . \quad (60)$$

whence

$$w^*(x, p + \kappa\mu_n^2) = \frac{h}{p + \kappa\mu_n^2} \cdot \frac{\exp\left[\dfrac{-x}{\sqrt{\kappa}} \sqrt{p + \kappa\mu_n^2}\right]}{h - \dfrac{1}{\sqrt{\kappa}} \sqrt{p + \kappa\mu_n^2}},$$

$$= \frac{h}{p + \kappa\mu_n^2} W_n^*(x, p), \quad . \quad . \quad . \quad . \quad (61)$$

since (60) is an identity in p and $k\mu_n^2 > 0$.

From the last equation we get

$$W_n^*(x, p) = \frac{1}{h} p w^*(x, p + \kappa\mu_n^2) + \frac{1}{h} \kappa\mu_n^2 w^*(x, p + \kappa\mu_n^2). \quad (62)$$

Consider the equation

$$\int_0^{\infty} \exp[-pt]w(x, t) \, dt = w^*(x, p), \quad . \quad . \quad (63)$$

which defines $w^*(x, p)$. Since this must be an identity in p, it follows that

$$w^*(x,\ p+\kappa\mu_n^2)=\mathrm{L}\{\exp\ [-\kappa\mu_n^2t]\,.\,w(x,\ t)\},\qquad . \quad (64)$$

or $\qquad \mathrm{L}^{-1}\{w^*(x,\ p+\kappa\mu_n^2)\}\exp\ [\kappa\mu_n^2t]\,.\,w(x,\ t)\ .\quad . \quad (65)$

Further

$$\mathrm{L}\left\{\ \frac{\partial}{\partial t}\ [\exp[-\kappa\mu_n^2t]w(x,t)]\ \right\}\ =p\mathrm{L}\{\exp\ [-\kappa\mu_n^2t]w(x,\ t)\}$$
$$=pw^*(x,\ p+\kappa\mu_n^2)\quad . \quad . \quad (66)$$

(from 20), and therefore

$$\mathrm{L}^{-1}\{pw^*(x,\ p+\kappa\mu_n^2)\}=\frac{d}{dt}\ \{\ \exp\ [\kappa\mu_n^2t]\,.\,w(x,\ t)\}.\quad (67)$$

In view of (61) and (67) the inversion of (62), *i. e.*, the result of operating on both members by the operator $\mathrm{L}^{-\prime}$ yields

$$\mathrm{W}_n(x,\ t)=\frac{1}{h}\ \frac{\partial}{\partial t}\ \{\exp[-\kappa\mu_n^2t]\,.\,w(x,\ t)\}$$
$$+\frac{1}{h}\ \kappa\mu_n^2\,.\ \exp\ [-\kappa\mu_n^2t]\,.\,w(x,\ t),$$
$$=\frac{1}{h}\ .\ \exp[-\kappa\mu_n^2t]\,.\ \frac{\partial}{\partial t}\ w(x,\ t).\quad : \quad . \quad . \quad . \quad . \quad (68)$$

In view of (68) our solution (59) becomes

$$\mathop{\mathrm{Lim}}_{a\to\infty}\ u(x,\ y,\ t)=\beta\ \sum_{n=0}^{\infty}\ \frac{\delta_n}{b}\ \cos\mu_n y\,.\,\exp\ [-\kappa\mu_n^2t]\,.\ \int_0^b\ \cos\mu_n\eta\ d\eta$$
$$\times\int_0^t\ \exp[-\kappa\mu_n^2\tau]\phi(\eta,\ \tau)\ \frac{\partial}{\partial\tau}\ w(x,\ t-\tau)\ d\tau,$$
$$=\beta\int_0^b\ d\eta\int_0^t\ \phi(\eta,\ \tau)\ \frac{\partial}{\partial t}\ w(x,\ t-\tau)$$
$$\times\Gamma_b(y,\ \eta,\ t-\tau)\ d\tau,\ .\quad . \quad . \quad (69)$$

where

$$\Gamma_b(y,\ \eta,\ t)=\frac{1}{b}\ \sum_{n=0}^{\infty}\ \delta_n\ \cos\mu_n y\,.\,\cos\mu_n\eta\,.\,\exp\ [-\kappa\mu_n^2t].$$

In the above derivation we assumed the function $w(x,\ t)$ known. Its expression can be obtained at once from formula (36), L 2, by putting there $f(x)=\phi(x,t)=0,\ \phi(t)=1$. That formula yields

$$w(x,\ t)=\frac{h}{2\sqrt{\pi\kappa}}\ \int_{\tau=0}^t\ \int_{\rho=0}^{\infty}\ (t-\tau)^{-\frac{3}{2}}\ (x+\rho)$$
$$\times\exp\left[\ \frac{-(x+\rho)^2}{4\kappa(t-\tau)}\ \right]\,.\,\exp\ [-h\rho]\,d\rho\,d\tau.\quad 70)$$

In (69) and (70) we have the complete solution of the system B when $a = \infty$. If in (69) we replace b and $\Gamma_b(y, \eta, t)$ by ∞ and $\Gamma_\infty(y, \eta, t)$, from (85), we obtain the solution corresponding to the case where both a and b are infinite.

We now proceed to the solution of our system C. Following the method of L 1, the Laplace transform of $v(x, y, t)$ is readily seen to satisfy the system

$$F \begin{cases} [\kappa(\partial^2/\partial x^2 + \partial^2/\partial y^2) - p]v^*(x, y, p) \\ \qquad\qquad\qquad = -f(x, y) - \psi^*(x, y, p), \quad (71) \\ (\alpha_1 \partial/\partial x + \beta_1)v^*(x, y, p) = 0 \quad \text{for } x = 0, \quad . \quad . \quad . \quad (72) \\ (\alpha_2 \partial/\partial x + \beta_2)v^*(x, y, p) = 0 \quad \text{for } x = a, \quad . \quad . \quad . \quad (73) \\ \dfrac{\partial}{\partial y}v^*(x, y, p) = 0 \quad \text{for } y = 0 \text{ and } y = b. \quad . \quad . \quad . \quad (74) \end{cases}$$

The general solution of F is given by

$$\begin{aligned} v^*(x, y, p) &= \int_0^b \int_0^a G(x, y, \xi, \eta)F(\xi, \eta)\,d\xi\,d\eta \\ &\quad + \int_0^b \int_0^a G(x, y, \xi, \eta)\psi(\xi, \eta, p)\,d\xi\,d\eta, \\ &= v_1^*(x, y, p) + v_2^*(x, y, p) \quad \text{(say)}, \quad . \quad . \quad . \quad (75) \end{aligned}$$

where $G(x, y, \xi, \eta)$ is the Green function corresponding to the homogeneous differential equation

$$(\partial/\partial x^2 + \partial/\partial y^2 + q^2)v^*(x, y, p) = 0 \quad . \quad . \quad . \quad (76)$$

in conjunction with the boundary conditions (73) and (74) where $p = -\kappa q^2$. Starting with the particular solution $(A \sin \lambda x + B \cos \lambda x) \cos \mu y$ where $\lambda^2 + \mu^2 = q^2$, the normalized orthogonal functions of the system (76), (73), and (74) are obtained in the form

$$\phi_{n, i}(x, y) = \phi_i(x) . \sqrt{\frac{2}{b}} \cos \mu_n y, \text{ where } \mu_n = \frac{n\pi}{b}, \quad (77)$$

$$\phi_i(x) = N(\lambda_i)(\beta_1 \sin \lambda_i x - \alpha_1 \lambda_i \cos \lambda_i x), \quad . \quad . \quad (78)$$

where the λ_i's are the roots of the transcendental equation

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\lambda \cos \lambda a - (\alpha_1\alpha_2\lambda^2 + \beta_1\beta_2) \sin \lambda a = 0 \quad (79)$$

and

$$\left(\frac{1}{N(\lambda_i)}\right)^2 = \int_0^a (\beta_1 \sin \lambda_i x - \alpha_1 \lambda_i \cos \lambda_i x)^2\,dx. \quad . \quad (80)$$

If we assume the validity of the bilinear expansion formula, we have

$$G(x, y, \xi, \eta) = \Sigma \frac{\phi_{n,i}(x, y)\phi_{n,i}(\xi, \eta)}{\kappa p + q_{n,i}^2}, \quad . \quad . \quad (81)$$

where
$$q_{n,i}^2 = \lambda_i^2 + \mu_n^2,$$

and the complete solution of the system C is ultimately obtained in the form

$$v(x, y, t) = \frac{1}{b} \sum_{n=0}^{\infty} \Sigma \delta_n \phi_i(x) \cos \mu_n y \cdot \exp[-\kappa(\lambda_i^2 + \mu_n^2)t)]$$

$$\times \int_0^a \int_0^b \phi_i(\xi) \cos \mu \eta \cdot f(\xi, \eta) \, d\xi \, d\eta$$

$$+ \frac{1}{b} \sum_{n=0}^{\infty} \Sigma \delta_n \phi_i(x) \cos \mu_n y \cdot \exp[-\kappa(\lambda_i^2 + \mu_n^2)t]$$

$$\times \int_0^a \int_0^b \phi_i(\xi) \cos \mu_n \eta \, d\eta \int_0^t \psi(\xi, \eta, \tau)$$

$$\times \exp[\kappa(\lambda_i^2 + \mu_n^2)\tau \, d\tau,$$

$$= v_1(x, y, t) + v_2(x, y, t) \quad \text{(say)}. \quad . \quad . \quad . \quad (82)$$

The first double sum in (82) is evidently the contribution arising from the initial temperature distribution, while the second sum arises from the distribution of heat generating sources.

Case $b = \infty$.

Consider first the contribution $v_1(x, y, t)$ arising from the initial temperature distribution. This may conveniently be rewritten in the form

$$v_1(x, y, t) = \Sigma \phi_i(x) \cdot \exp[-\kappa \lambda_i^2 t]$$
$$\times \int_0^a \phi_i(\xi) \, d\xi \int_0^b f(\xi, \eta) \, \Gamma_b(y, \eta, t) \, d\eta, \quad (83)$$

where we have put

$$\Gamma_b(y, \eta, t) = \frac{1}{b} \sum_{n=0}^{\infty} \cos \mu_n y \cdot \cos \mu_n \eta \, \delta_n \cdot \exp[-\kappa \mu_n^2 t], (84)$$

the subscript in the first member being a reminder that the expansion in the second member depends on b.

Comparing with formula (23) in L 2, we have

$$\underset{b \to \infty}{\text{Lim}} \, \Gamma_b(y, \eta, t) = \frac{1}{2\sqrt{\pi \kappa t}} \left\{ \exp\left[\frac{-(y-\eta)^2}{4\kappa t} \right] \right.$$
$$\left. + \exp\left[\frac{-(y+\eta)^2}{4\kappa t} \right] \right\} = \Gamma_\infty(y, \eta, t) \quad \text{(say)}. \quad (85)$$

In view of (83) and (85) we finally get

$$\operatorname*{Lim}_{b \to \infty} v_1(x, y, t) = \frac{1}{2\sqrt{\pi\kappa t}} \Sigma\phi_i(x) . \exp\left[-\kappa\lambda_i^2 t\right] . \int_0^a \phi_i(\xi) \, d\xi$$
$$\times \int_0^\infty f(\xi, \eta) \left\{ \exp\left[\frac{-(y+\eta)^2}{4\kappa t}\right] + \exp\left[\frac{-(y-\eta)^2}{4\kappa t}\right] \right\} d\eta.$$
$$\qquad\qquad . \quad . \quad . \quad (86)$$

A similar treatment applied to the contribution $v_2(x, y, t)$ yields

$$\operatorname*{Lim}_{b \to \infty} v_2(x, y, t) = \frac{1}{2\sqrt{\pi\kappa}} \Sigma\phi_i(\lambda) . \exp\left[-\kappa\lambda_i^2 t\right]$$
$$\times \int_0^a \phi_i(\xi) \, d\xi \int_0^\infty d\eta . \int_0^t \psi(\xi, \eta, \tau)$$
$$\times \frac{1}{\sqrt{t-\tau}} \left\{ \exp\left[\frac{-(y+\eta)^2}{4\kappa(t-\tau)}\right] \right.$$
$$\left. + \exp\left[\frac{-(y-\eta)^2}{4\kappa(t-\tau)}\right] \right\} d\tau. \quad . \quad . \quad . \quad (87)$$

Case $a = \infty$.

The expression for $v_1(x, y, t)$ may now conveniently be rewritten in the form

$$v_1(x, y, t) = \sum_{n=0}^\infty \cos\mu_n y . \exp\left[-\kappa\mu_n^2 t\right]$$
$$\times \int_0^b \cos\mu_n . \theta_a(x, t, \eta) \, d\eta, \quad (88)$$

where

$$\theta_a(x, \eta, t) = \int_0^a f(\xi, \eta) \Sigma\phi_i(x)\phi_i(\xi) . \exp\left[-\kappa\lambda_i^2 t\right] d\xi, \quad (89)$$

and the problem reduces to calculating

$$\operatorname*{Lim}_{a \to \infty} \theta_a(x, \eta, t).$$

Consider the auxiliary problem in heat conduction defined by the system

$$\text{G} \begin{cases} \kappa\dfrac{\partial^2}{\partial x^2}\omega(x, \eta, t) = \dfrac{\partial}{\partial t}\omega(x, \eta, t), \\[2mm] \operatorname*{Lim}_{t \to 0}\omega(x, t) = f(x, \eta), \\[2mm] \alpha_1\dfrac{\partial\omega}{\partial x} + \beta_1\omega = 0 \quad \text{for } x = 0, \\[2mm] \alpha_2\dfrac{\partial\omega}{\partial x} + \beta_2\omega = 0 \quad \text{for } x = a. \end{cases}$$

In G we have the uni-dimensional counterpart of the system C for the case $\psi(x, y, t) = 0$, the initial temperature function $f(x, y)$ of the latter system being replaced by $f(x, \eta)$ where η is a fixed parameter. The solution of the system G obtained in a manner entirely similar to that of C may be written down at once in the form

$$\omega(x, \eta, t) = \int_0^a f(\xi, \eta) \Sigma \phi_i(x) \phi_i(\xi) . \exp[-\kappa \lambda_i^2 t] d\xi, \quad (90)$$

where the ϕ's have the same significance as above (equation (78)) and the summation extends over the roots of (79).

From (89) and (90) it follows that

$$\theta_\infty(x, \eta, t) = \underset{a \to \infty}{\mathrm{Lim}}\ \omega(x, \eta, t) . \quad . \quad . \quad . \quad (91)$$

It is easily seen that as $a \to \infty$ all the roots of (79) tend to 0 irrespectively of the numerical values of the constants α and β. It follows that as $a \to \infty$, $\omega(x, t, \eta)$ tends to the solution of the system G$'$ obtained from G by replacing the boundary conditions at a by the new boundary condition

$$\underset{a \to 0}{\mathrm{Lim}}\ \omega(x, t, \eta) = 0 \quad . \quad . \quad . \quad : \quad . \quad (92)$$

and by replacing the interval 0 to a by the interval 0 to ∞. Clearly the solution of G$'$ identical with the function $\theta_\infty(x, t, \eta)$ is nothing more than the temperature distribution in a semi-infinite solid initially at temperature $f(x, \eta)$ when the boundary condition at $x = 0$ is

$$\alpha_1 \frac{\partial \omega}{\partial x} + \beta_1 \omega = 0. \quad . \quad : \quad \quad . \quad . \quad (93)$$

The last equation may be written in the form $\dfrac{\partial \omega}{\partial x} = h\omega$ where $h = \dfrac{-\beta_1}{\alpha_1}$, showing that the boundary condition at $x = 0$ corresponds to radiation into a medium at $0°$. We thus arrive at the conclusion that $\theta_\infty(x, t, \eta)$ is the temperature distribution in a semi-infinite solid initially at temperature $f(x, \eta)$ radiating at $x = 0$ into a medium

at $0°$. This is a well-known problem, the solution of which is †

$$\theta_\infty(x, t, \eta) = \frac{1}{2\sqrt{\pi\kappa t}} \int_0^\infty \left\{ \exp\left[\frac{-(x-\rho)^2}{4\kappa t}\right] + \exp\left[-\frac{(x+\rho)^2}{4\kappa t}\right] \right.$$

$$\left. -2h \int_0^\infty \exp\left[-h\xi - \frac{(x+\rho+\xi)^2}{4\kappa t}\right]d\xi \right\} f(\rho, \eta)d\rho). \quad (94)$$

In (88) and (94) we have the complete solution for $v_1(x, y, t)$ for the case $a = \infty$.

In an entirely similar fashion we may derive the expression for $\operatorname*{Lim}_{a \to \infty} v_2(x, y, t)$. As a matter of fact, the desired expression may be obtained from that of $z_1(x, y, t)$ by replacing t and $f(x, \eta)$ by $(t-\tau)$ and $\psi(x, \eta, \tau)$ and integrating between o and t. Thus collecting our results we have

$$\operatorname*{Lim}_{a \to \infty} v(x, y, t)$$

$$= \frac{1}{2\sqrt{\pi\kappa t}} \int_{\eta=0}^b \int_{\rho=0}^\infty \left\{ \exp\left[-\frac{(x-\rho)^2}{4\kappa t}\right] + \exp\left[-\frac{(x+\rho)^2}{4\kappa t}\right] \right.$$

$$\left. -2h \int_0^\infty \exp\left[-h\xi - \frac{(x+\xi+\rho)^2}{4\kappa t}\right]d\xi \right\} \Gamma_b(y, \eta, t)$$

$$\times f(\rho, \eta) \, d\rho \, d\eta$$

$$+ \frac{1}{2\sqrt{\pi\kappa}} \int_{\eta=0}^b \int_{\rho=0}^\infty \int_{\tau=0}^t (t-\eta)^{-\frac{1}{2}} \left\{ \exp\left[-\frac{(x-\rho)^2}{4\kappa(t-\tau)}\right] \right.$$

$$+ \exp\left[\frac{-(x+\rho)^2}{4\kappa(t-\tau)}\right]$$

$$\left. -2h \int_0^\infty \exp\left[-h\xi - \frac{(x+\xi+\rho)^2}{4\kappa(t-\tau)}\right]d\xi \right\} \Gamma_b(y, \eta, t-\tau)$$

$$\times \psi(\rho, \eta, \tau) \, d\rho \, d\eta \, d\tau, \quad . \quad . \quad . \quad . \quad (95)$$

where $\Gamma_b(y, \eta, t)$ is given by (84). If in (95) we replace b by ∞ and $\Gamma_b(y, \eta, t)$ by $\Gamma_\infty(y, \eta, t)$, we obtain the complete solution for the case $a = b = \infty$.

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† See Carslaw, Article 83.

XLII. *A new Family of Wing Profiles.* *By* N. A. V. PIERCY, *D.Sc.*, *M.Inst.C.E.*, R. W. PIPER, *M.Sc.*, *and* J. H. PRESTON, *Ph.D.**

1. *Introduction.*

RECENT improvement in the efficiency of aeroplanes has renewed interest in the question of shaping their contours mathematically. To some extent this process is already followed, if only to avoid sudden changes in curvature, arbitrary formulæ often being employed. But in the case of wings and similar parts which extend considerably across the direction of motion, it is desirable to make use of the method of conformal transformation, so that the irrotational flow outside the boundary layers of regions remote from the ends may be calculated. This method has long been appreciated in aeronautics, but without great complication application has so far involved crudities in the shapes obtained that are incompatible with efficiency, particularly at high speeds.

Since conformally transformed aerofoils were introduced by Joukowski in 1910, frequent attempts have been made to improve their shapes the better to meet aerodynamic and structural requirements of aeroplane wings. A more recent series of transformed profiles, in common use for theoretical investigations, is the Kármán-Trefftz, a family which avoids the cusp originally existing at the trailing edge and permits the maximum thickness of the streamline section to be located farther back than formerly, as appears generally to be desirable. In the thin cambered sections used for wings, the characteristic shape of this series suffers from two disadvantages from the point of view of practical utilization, viz., rather too sharp a nose and a certain concavity in the undersurface towards the tail. But another objection to this and other families is the loss of the simplicity of Joukowski's early transformation. Graphical tracing becomes impracticable and calculation laborious. The complication is felt in theoretical investigations in which the precise shape of contour may be of secondary importance, although cusped profiles would be prohibited.

In the new family developed in the present paper the foregoing disadvantage—and some others—can be avoided. The profiles are quickly calculated, or may be drawn by mechanical means if desired, and they include cambered sections of bi-convex type closely resembling those of successful modern wings.

The essential steps of the analysis can be described briefly. In view of timely interest, however, we have added some notes on the properties of thin shapes and other matters of importance in aeronautical applications.

2. *General Equation for the Transformed Profile.*

The starting point is the family of limaçons, $r = a(b + \cos\theta)$, the inner loops of which family include a system of suitable *symmetrical* profiles. Now the inverse

Fig. 1.
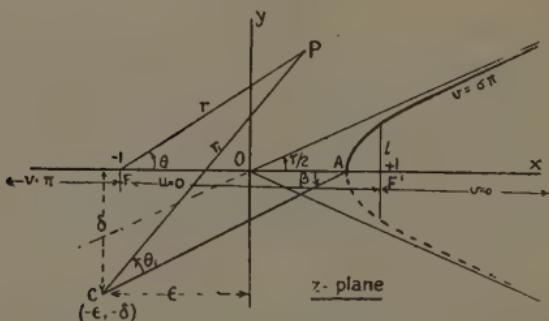
curves of a family of confocal hyperbolas with respect to a focus can be written in the form $lr = e\cos\theta - 1$. Hence an appropriate choice of constants allows an inner loop of any curve of the former type to be regarded as the inverse of one branch of a particular hyperbola with respect to the external focus. If, instead of the external focus F, a point C near to F be chosen as the centre of inversion, an asymmetric wing profile will be expected to result.

Let

$$z = x + iy = \cosh w, \quad \ldots \quad (1)$$

where $w = u + iv$, and u may assume any real value but v is restricted to the range $\pi \geqslant v \geqslant 0$. Then $v = \text{const.} = \sigma\pi = \tau/2$ defines one branch of the hyperbola

$$x^2/\cos^2\sigma\pi - y^2/\sin^2\sigma\pi = 1, \quad \ldots \quad (2)$$

and, if $\frac{1}{2} > \sigma \geqslant 0$, this branch lies on the positive side

of the y-axis. The foci F′, F (fig. 1) are at ± 1, 0, and the vertex A is at $\cos \sigma\pi$, 0.

Referred to F $(-1, 0)$, the equation in polar coordinates of the hyperbola is

$$l/r = \text{e} \cos \theta - 1, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where l is the semi-latus rectum and e the eccentricity, given by

$$\text{e} = \sec \tfrac{1}{2}\tau, \quad l = \text{e} - 1/\text{e} \quad . \quad . \quad . \quad . \quad (4)$$

$\tfrac{1}{2}\tau$ is marked in the figure and will be found to have a special significance in the transformed plane.

Let C $(-\epsilon, -\delta)$ be the chosen centre of inversion. It is convenient to introduce polar coordinates r_1, θ_1, referred to C and CA. From fig. 1 we have immediately

$$r_1 \sin (\theta_1 + \beta) - \delta = r \sin \theta,$$
$$r_1 \cos (\theta_1 + \beta) - (\epsilon - 1) = r \cos \theta,$$

Fig. 2.

z_2 - plane
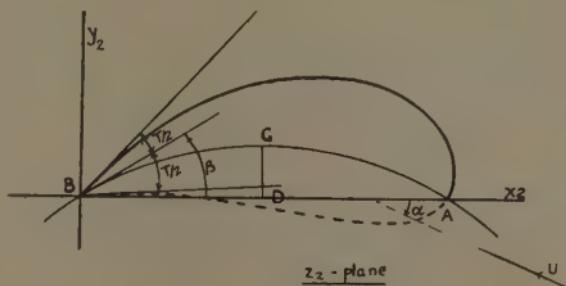
and (3) becomes

$$\{l - \text{e}\,[r_1 \cos (\theta_1 + \beta) - (\epsilon - 1)]\}^2$$
$$= [r_1 \sin (\theta_1 + \beta) - \delta]^2 + [r_1 \cos (\theta_1 + \beta) - (\epsilon - 1)]^2.$$
$$\quad . \quad . \quad . \quad (5)$$

We proceed to carry out the inversion and reflexion in the line AC, defined by the transformation

$$z_1 z_2 = e^{i\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6$$

(which gives $r_2 = 1/r_1$, $\theta_2 = -\theta_1$), obtaining from (5)

$$r_2{}^2[l^2 - \delta^2 + l\text{e}(\epsilon^2 - 1)]$$
$$- 2r_2[l\text{e}\epsilon \cos (\theta_2 - \beta) + \delta \sin (\theta_2 - \beta)]$$
$$+ \text{e}^2 \cos^2 (\theta_2 - \beta) - 1 = 0. \quad . \quad . \quad (7)$$

This equation may be solved in chosen circumstances for r_2 in terms of θ_2, whence the transformed profile

2 F 2

is obtained in the z_2-plane (see fig. 2).   This profile will be called the aerofoil.

Transforming the initial line CA, we have, for $\theta_2 = 0$,

$$\frac{1}{\text{BA}} = \frac{\epsilon + \cos \sigma\pi}{\cos \beta} = \frac{1}{c}, \text{ say, } \quad . \quad . \quad . \quad (8)$$

suggesting the introduction of non-dimensional lengths r_2/c, etc.

The radii through C parallel to the asymptotes in fig. 1 meet the hyperbola at infinity, whence these lines transform into the tangents at the tail B of the aerofoil. Hence the "tail angle" of the aerofoil $= \tau$, *i. e.*, it is equal to the angle between the asymptotes of the hyperbola chosen.

If C, the centre of inversion, lie on an asymptote produced, or between the asymptotes produced, the aerofoil contour will lie entirely between the tangents at B, and will be everywhere convex to B.   But if C lie outside the asymptotes produced, the profile in the z_2-plane will be concavo-convex.   Thus a suitable location of the centre of inversion gives either a bi-convex or a concavo-convex aerofoil, as desired.   This generalization as to shape constitutes a distinguishing feature of the present family.

The axis of the hyperbola transforms into an arc of a circle, BGA, of which the camber $\text{GD}/\text{BA} = \frac{1}{2} \tan \frac{1}{2}\beta$.

3. *Two Special Families.*

The simplest case occurs when $\epsilon = 1$ and $\beta = 0 = \delta$, C coinciding with F.   Either from (7) or directly we then have, using (8),

$$\frac{r_2}{c} = \frac{\cos \theta_2 - \cos \sigma\pi}{1 - \cos \sigma\pi}. \quad . \quad . \quad . \quad . \quad (9)$$

This represents the series of symmetrical profiles illustrated in fig. 3, the inner loops of the limaçon family, which are suitable for tail planes, fins, and strut sections in aircraft.

Keeping $\epsilon = 1$, but now allowing β to vary from zero, still greatly simplifies (7), and, the ensuing shapes being of interest, the result may be noted:

$$r_2{}^2(l^2 - \delta^2) - 2r_2[le \cos(\theta_2 - \beta) + \delta \sin(\theta_2 - \beta)]$$
$$+ e^2 \cos^2(\theta_2 - \beta) - 1 = 0.$$
$$. \quad . \quad . \quad (10)$$

To obtain bi-convex shapes we must have

$$\tan \beta < \tan \tfrac{1}{2}\tau/(1+\cos \tfrac{1}{2}\tau)$$

or $$\beta < \tfrac{1}{4}\tau, \qquad \qquad \qquad \Big\}\quad . \quad . \quad . \quad (11)$$

approximately, for thin sections.

This second family is illustrated in fig. 4, and thinner members are seen to conform to the requirements of aeroplane wings intended for high speeds.

Fig. 3.
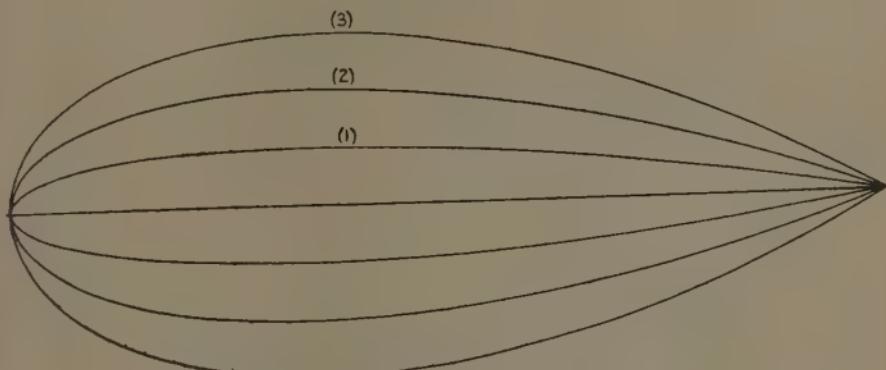
Three symmetrical profiles ($\epsilon=1$, $\beta=0$).

Fig. 4.
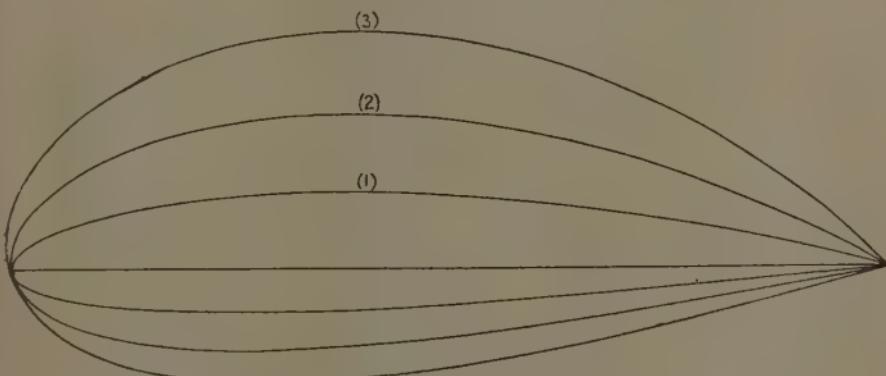
Three wing profiles ($\epsilon=1$, $\beta=\tau/4=\sigma\pi/2$).

4. *Range of Shape of Symmetric Aerofoils.*

Two simple investigations illustrate further the possible variation in shape of contour and provide a guide in arranging for a particular shape to meet other conditions. The calculations are carried out for symmetrical aerofoils.

(*a*) Keeping $\epsilon=1$, we investigate the relation between

the distance x from the tail at which maximum thickness occurs and the tail angle $\tau = 2\sigma\pi = 2\sec^{-1}e$.

Dropping suffixes as unnecessary in the present section, and taking as the equation of the profile $lr = e\cos\theta - 1$, where l and e are defined as in (4), we have for a maximum ordinate

$$l\frac{d}{d\theta}(r\sin\theta) = 2e\cos^2\theta - \cos\theta - e = 0,$$

TABLE I.

$\tau/2$.	0.	$\pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$e = \sec\frac{1}{2}\tau$	1	$2\sqrt{3}/3$	$\sqrt{2}$	2	∞
x/c	0·667	0·642	0·613	0·579	0·500

Fig. 5.
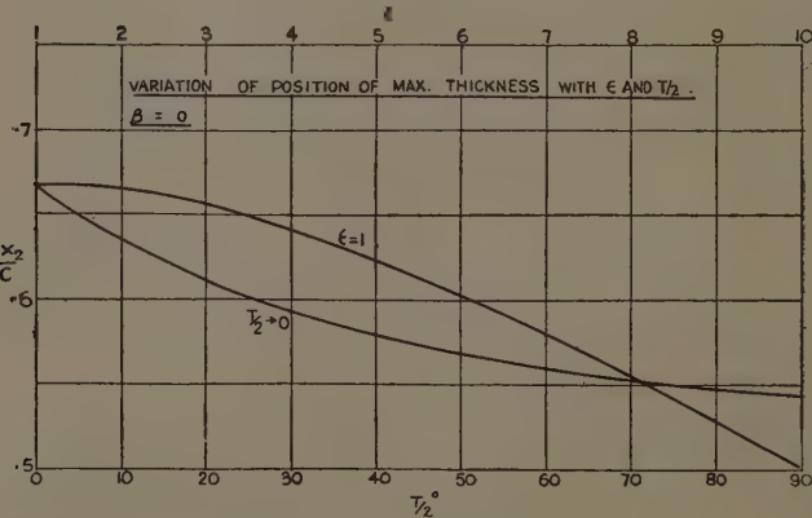
i. e.,
$$\cos\theta = \frac{(1 + \sqrt{1 + 8e^2})}{4e},$$

whence
$$lr\cos\theta = \frac{1 + \sqrt{(1 + 8e^2)}}{4e} \cdot \frac{\sqrt{(1 + 8e^2)} - 3}{4}.$$

Finally,
$$\frac{x}{c} = \frac{lr\cos\theta}{lc} = \frac{4e^2 - 1 - \sqrt{(1 + 8e^2)}}{8e(e - 1)} . \quad . \quad . \quad (12)$$

Some values are given in Table I. These lie on the curve marked $\epsilon = 1$ in fig. 5, which shows that the maximum

ordinate of thin sections is located at $c/3$ behind the nose when $\epsilon = 1$.

(*b*) The above positions for the maximum thickness may be changed by varying ϵ from unity This is readily shown in detail for the thin sections of practical interest. It is necessary to return to suffixes.

We have

$$\left.\begin{array}{l} x_1 = \epsilon + \cosh u \cos \sigma\pi, \\ y_1 = \sinh u \sin \sigma\pi, \\ y_2 = y_1/r_1{}^2, \end{array}\right\} \quad . \quad . \quad . \quad (13)$$

and for maximum y_2

$$\frac{dy_2}{du} = \frac{1}{r_1{}^4}\left[(x_1{}^2 - y_1{}^2)\frac{dy_1}{du} - 2x_1 y_1 \frac{dx_1}{du}\right] = 0$$

or

$$\left(x_1{}^2 - y_1{}^2 - \frac{2x_1 y_1{}^2}{x_1 - \epsilon}\cot^2 \sigma\pi\right)\frac{dy_1}{du} = 0,$$

and this can be arranged as

$$x_1{}^2 - \sin^2 \sigma\pi\left[\left(\frac{x_1 - \epsilon}{\cos \sigma\pi}\right)^2 - 1\right] - \frac{2x_1}{x_1 - \epsilon}\left[(x_1 - \epsilon)^2 - \cos^2 \sigma\pi\right] = 0$$

$$. \quad . \quad . \quad (14)$$

on expressing y_1 in terms of x_1 by (13), since $\sigma\pi \neq 0$.

For thin symmetrical aerofoils $\sigma\pi$ is small and $x_2 = 1/x_1$. Hence, by (8), the chord $c = 1/(1+\epsilon)$, whilst (14) approximates to

$$x_1{}^3 - 3x_1{}^2\epsilon - 2x_1(1-\epsilon^2) = 0.$$

Thus, finally, the maximum ordinate occurs at

$$\frac{x_2}{c} = \frac{2(\epsilon+1)}{3\epsilon + \sqrt{(\epsilon^2+8)}} = \frac{3\epsilon - \sqrt{(\epsilon^2+8)}}{4(\epsilon-1)}. \quad . \quad (14\,a)$$

This function of ϵ is plotted as the curve marked $\frac{1}{2}\tau \rightarrow 0$ in fig. 5. A fairly wide variation in the position along the contour of the maximum thickness of the section is seen to be possible.

5. *Mechanical Description.*

The contours of the profiles, except in the neighbourhood of the tail, can be described mechanically by the use of Hart's linkage or the simple Peaucellier's cell (fig. 6). Near the tail the profile corresponds to parts

of the hyperbola at a large distance from its vertex. Since these approximate asymptotically to straight lines, the corresponding parts of the aerofoil contours can be drawn with fair accuracy as arcs of circles, the inverse of the asymptotes.

Fig. 6.

(a)

(b)
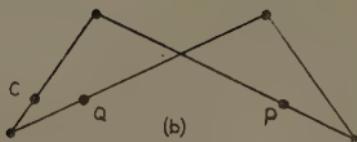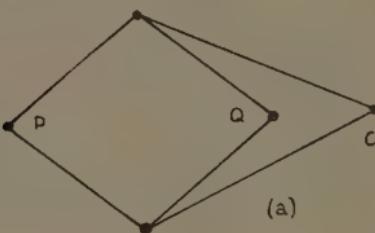
(a) Peaucellier's, (b) Hart's linkage.
C, centre of inversion ; P, Q, tracing points.

6. *Transformation of the Profile into a Circle.*

Having now discussed the triply infinite family of aerofoils in respect of shape and ease of manipulation, we proceed to the investigation necessary to establish their place in Aerofoil Theory. The first problem is to transform the profile into a circle, the region external to the profile mapping into that external to the circle.

Instead of dealing with (7), it is convenient to retain the complex variables, and then we have, in the general case, that the branch of the hyperbola

$$z_1 = \epsilon + i\delta + \cosh w \quad . \quad . \quad . \quad . \quad (15)$$

transforms into the profile

$$z_2 = \frac{e^{i\beta}}{\epsilon + i\delta + \cosh w} \quad . \quad . \quad . \quad . \quad (16)$$

by the inversion (6). The transformation to a circle will be carried out by a series of steps which are indicated in fig. 7, and analytically are as follows.

Referring to the z-plane, fig. 7 (a), the hyperbola is given by $v=\sigma\pi$, the real axis to the left of the focus F by $v=\pi$, the real axis to the right of F' by $v=0$, and the part of the real axis between F and F' by $u=0$.

Fig. 7.
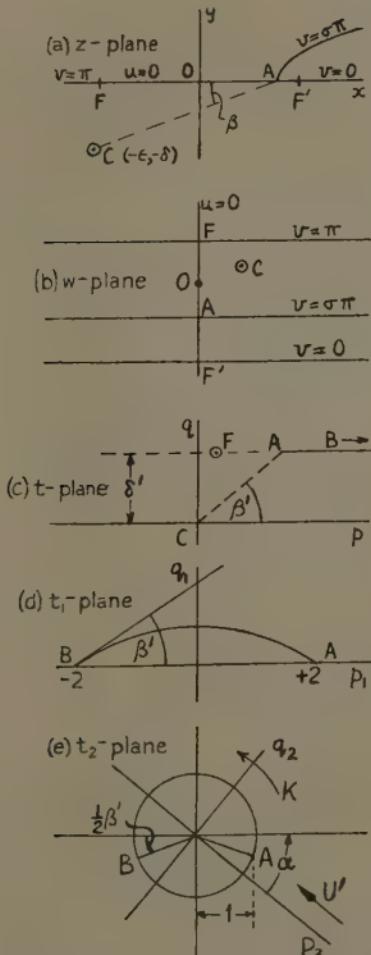
Turning to the w-plane, fig. 7 (b), the hyperbola is represented by the straight line $v=\sigma\pi$, and the axis of symmetry by the line $v=\pi$, together with that part AF of the line $u=0$ which lies between $v=\sigma\pi$ and $v=\pi$. In this plane the point C is represented by the complex u_c+iv_c given implicitly by

$$z' = -\epsilon-i\delta = \cosh w_c = \cosh (u_c+iv_c) \quad \bullet \quad \bullet \quad (17)$$

Now we transform to a t -plane whose complex coordinate is

$$t = p + iq \quad . \quad . \quad . \quad . \quad . \quad (18)$$

by the formula

$$t = \epsilon' + i\delta' + \cosh \frac{w - i\sigma\pi}{1 - \sigma} \cdot \quad . \quad . \quad . \quad (19)$$

In this formula

$$t' = -\epsilon' - i\delta' = \cosh \frac{u_c + i(v_c - \sigma\pi)_c}{1 - \sigma} = \cosh p_c . \quad (20)$$

represents the point corresponding to the centre of inversion. The formula (19) maps the infinite strip between $v = \sigma\pi$ and $v = \pi$ in the w -plane on to the whole of the t -plane (fig. 7 (c)). In this process the boundary $v = \sigma\pi$ becomes the part AB of a line, parallel to and at a distance δ' from the real axis, which stretches from the point $(1 + \epsilon' + i\delta')$ to ∞ .

The next step is to transform the boundary AB into a circular arc in a t_1 -plane (fig. 7 (d)). With β' defined as in the preceding plane (fig. 7 (c)), this is achieved by the transformation

$$t(t_1 + 2)e^{-i\beta'} = 4(1 + \epsilon') \sec \beta'. \quad . \quad . \quad (21)$$

The circular arc stretches between $p_1 = \pm 2$ in the t_1 -plane.

Finally, the transformation

$$t_1 = (t_2 e^{-i\alpha'} + i \tan \tfrac{1}{2}\beta') + 1/(t_2 e^{-i\alpha'} + i \tan \tfrac{1}{2}\beta') \quad . \quad (22)$$

maps the boundary into the circle $|t_2| = \sec \tfrac{1}{2}\beta'$ in the t_2 -plane (fig. 7 (e)), where α' is shown.

 α' will define generally the direction of the oncoming stream, whose flow round this circle in the presence of a circulation will be transformed to give the flow round the aerofoil. Hence a velocity U′ and circulation K (which is determined later by Joukowski's hypothesis) have been marked in this final figure. It is necessary first to show, however, that carrying out this process by the means which have been found is justifiable. The proof is indicated in the following section.

7. *The Mapping Function.*

It is known that the conformal transformation which maps the region external to the circle in the t_2 -plane

into the region external to the aerofoil in the z_2-plane must be of the type

$$z_2 = a_0 t_2 + a_1 + a_2/t_2 + a_3/t_2{}^2 + \ldots \quad . \quad . \quad (23)$$

It will now be shown that the sequence of transformations given in the preceding section satisfies this condition.

Considering the transformation (16), which is subject to the restriction $\pi \geqslant v \geqslant 0$, it is 1 to 1, and z_2 is an analytic function of w, with a simple pole at w_c, see (17).

Again, the transformation (21), subject to the same restriction as regards v, is 1 to 1, and t_1 is an analytic function of w, with a simple pole at w_c.

Hence z_2 is an analytic function of t_1, with a simple pole at $t_1 = \infty$.

With the restriction that t_2 lies outside the circle $|t_2| = \sec \frac{1}{2}\beta'$, the transformation (22) is 1 to 1, and t_1 is an analytic function of t_2, with a simple pole at $t_2 = \infty$.

Hence z_2 is an analytic function of t_2 outside the circle in the t_2-plane, with a simple pole at ∞ .

From (15), (17), (19), and (20)

$$z_1 + z_1{}' = \cosh w,$$
$$t + t' = \cosh \frac{w - i\sigma\pi}{1 - \sigma},$$

and

$$z_1 + z' \pm \sqrt{(z_1 + z')^2 - 1} = e^{\pm w}$$
$$= e^{\pm i\sigma\pi}\{t + t' \pm \sqrt{(t + t')^2 - 1}\}^{1 - \sigma},$$

where either all the $+$ signs or all the $-$ signs are taken together. Hence

$$2(z_1 + z') = e^{i\sigma\pi}\{t + t' + \sqrt{(t + t')^2 - 1}\}^{1 - \sigma}$$
$$+ e^{-i\sigma\pi}\{t + t' - \sqrt{(t + t')^2 - 1}\}^{1 - \sigma}.$$

The expression on the right may be expanded for $t' \neq 1$ by the Binomial Theorem for sufficiently small values of $|t|$, and, after collecting terms, expanded again. A little reduction then gives

$$2(z_1 + z') = 2 \cosh w_c + 2c_1 t + 2c_2 t^2 + \ldots,$$

where

$$c_1 = (1 - \sigma) \sinh w_c / \sinh p_c,$$
$$c_2 = (1 - \sigma)[\sinh (p_c - w_c) - \sigma \cosh w_c \sinh p_c]/2 \sinh^3 p_c,$$

$$c_3 = (1-\sigma)[\tfrac{1}{3}\sigma(\sigma+1)\sinh w_c \sinh^2 p_c$$
$$-\cosh p_c \sinh (p_c - w_c)$$
$$+\sigma \sinh p_c \cosh (p_c - w_c)]/2 \sinh^5 p_c,$$

with the notation indicated in (20). Hence for sufficiently small values of $|t|$

$$z_1 = c_1 t + c_2 t^2 + c_3 t^3 + \ldots \quad . \quad . \quad . \quad (24)$$

From (22)

$$t_1 + 2 = (t_2 e^{-i\alpha'} + i \tan \tfrac{1}{2}\beta' + 1)^2 / (t_2 e^{-i\alpha'} + i \tan \tfrac{1}{2}\beta'),$$

and from (21)

$$t/\lambda = 1/(t_1 + 2), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where

$$\lambda = 4(1+\epsilon') \sec \beta' e^{i\beta'} = 4(1-\cosh p_c). \quad . \quad . \quad (26)$$

Thus, introducing for brevity

$$\mu = i \tan \tfrac{1}{2}\beta' + 1 = e^{i\frac{1}{2}\beta'} \sec \tfrac{1}{2}\beta', \quad . \quad . \quad . \quad (27)$$

we have

$$t = \frac{\lambda}{t_2 e^{-i\alpha'} + \mu} - \frac{\lambda}{(t_2 e^{-i\alpha'} + \mu)^2}. \quad . \quad . \quad . \quad (28)$$

Substituting for t from (28) in (24), and expanding by the Binomial Theorem for sufficiently large values of $|t_2|$, we find

$$z_1 = b_1/t_2 + b_2/t_2{}^2 + b_3/t_2{}^3 + \ldots, \quad . \quad . \quad (29)$$

where

$$\left.\begin{aligned}
b_1 &= \lambda c_1 e^{i\alpha'}, \\
b_2 &= -\lambda e^{2i\alpha'}[c_1(\mu+1) - \lambda c_2], \\
b_3 &= \lambda e^{3i\alpha'}[c_1\mu(\mu+2) - 2c_2\lambda(1+\mu) + c_3\lambda^2].
\end{aligned}\right\} \quad (30)$$

Hence, finally,

$$\left.\begin{aligned}
z_2 = \frac{e^{i\beta}}{z_1} &= e^{i\beta}\frac{t_2}{b_1}\left(1 + \frac{b_2}{b_1 t_2} + \frac{b_3}{b_1 t_2{}^2} + \ldots\right)^{-1} \\
&= e^{i\beta}\left(\frac{t_2}{b_1} - \frac{b_2}{b_1{}^2} + \frac{b_2{}^2 - b_1 b_3}{b_1{}^3 t_2} + \ldots\right) \\
&= a_0 t_2 + a_1 + a_2/t_2 + a_3/t_2{}^2 + \ldots
\end{aligned}\right\} \quad (31)$$

for sufficiently large values of $|t_2|$.

A power series of this form has a singularity on its circle of convergence; $z_2 - a_0 t_2$ is analytic, and has no pole outside the circle $|t_2| = \sec \tfrac{1}{2}\beta'$; hence the series is analytic outside this circle.

Since at infinity $dz_2/dt_2 = a_0$, and in general a_0 is not equal to unity, the flows in the z_2- and t_2-planes are not in general equal. This may be avoided by introducing a suitable constant into one of the transformations, if desired. Alternatively, the undisturbed velocities are easily related to one another, as in the next section.

8. (i.) *Relation between the Planes of the Aerofoil and Circle.*

The utility of the system which has been described depends upon the existence of simple formulæ of transformation between the z_2- and t_2-planes, with means of easy reference to the plane of the hyperbola. In the present section we consider the relations between the velocities at infinity.

The velocity U at infinity in the plane of the aerofoil is derived from U′, that in the plane of the circle by

$$\mathrm{U} = \mathrm{U}' \left| \frac{dt_2}{dz_2} \right|_\infty = \mathrm{U}' \left| \frac{1}{a_0} \right|, \quad . \quad . \quad . \quad (32)$$

from (23). Hence, from (31), (30), and (26),

$$\mathrm{U} = \mathrm{U}' \mid b_1 e^{-i\beta} \mid$$
$$= 4\mathrm{U}'(1-\sigma)(1+\epsilon') \sec \beta' \left| \frac{\sinh w_c}{\sinh p_c} \right| . \quad . \quad . \quad (33)$$

In the particular case of symmetric aerofoils this expression reduces to the simple result

$$\mathrm{U} = 8\mathrm{U}'(1-\sigma)^2 . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Proceeding in a similar way to relate angles in the two planes, we have, α being associated with the aerofoil and α' with the circle,

$$\alpha = \arg (b_1 e^{-i\beta})$$
$$= \alpha' + \beta' - \beta + \arg \left(\frac{\sinh w_c}{\sinh p_c} \right). \quad . \quad . \quad (35)$$

9. (ii.) *Relation between Coordinates.*

This section collects the formulæ necessary for determining corresponding points in the two planes, particularly on the boundaries: the aerofoil and the circle.

Dealing first with the aerofoil, if we write

$$\mathrm{X} + i\mathrm{Y} = 1/z_2 = z_1 e^{-i\beta},$$

then from (13) we shall have

$$x_2 = X/(X^2+Y^2), \quad y_2 = -Y/(X^2+Y^2). \qquad (36)$$

It is easily found that generally

$$\left.\begin{array}{l} X = \epsilon \cos\beta + \delta \sin\beta + \mathrm{R.P.}[\cosh(u+iv)e^{-i\beta}], \\ Y = -\epsilon \sin\beta + \delta \cos\beta + \mathrm{I.P.}[\cosh(u+iv)e^{-i\beta}]. \end{array}\right\} \qquad (37)$$

On the boundary $v = \sigma\pi$, and these formulæ take the convenient forms

$$X = \epsilon \cos\beta + \delta \sin\beta + \tfrac{1}{2}[e^u \cos(\sigma\pi-\beta) + e^{-u} \cos(\sigma\pi+\beta)],$$
$$Y = -\epsilon \sin\beta + \delta \cos\beta + \tfrac{1}{2}[e^u \sin(\sigma\pi-\beta) - e^{-u} \sin(\sigma\pi+\beta)].$$
$$\qquad\qquad (38)$$

Hence, for chosen values of the parameter u the coordinates of points on the aerofoil can be calculated readily, and expressed in non-dimensional form by (8).

To find corresponding points on the circle, we proceed as follows.

One expression for t_1 (in terms of t_2) is given by (22), another (in terms of t) is given by (21), equating these leads, with the notation of (26) and (27), to

$$\frac{(t_2 e^{-i\alpha'}+\mu)^2}{t_2 e^{-i\alpha'}+\mu-1} = \frac{\lambda}{i}. \qquad (39)$$

Now on the boundary $t_2 = \sec\tfrac{1}{2}\beta' \cdot e^{i\theta_2}$ and $v = \sigma\pi$. Substitution and reduction give

$$\frac{i\delta'}{1+\epsilon'} + f(u) = 2\frac{\cos\tfrac{1}{2}\beta'}{\cos\beta'} \cdot \frac{e^{i\frac{1}{2}\beta'}+i\sin\tfrac{1}{2}\beta'e^{-i\chi}}{1+\cos\chi}, \qquad (40)$$

where

$$f(u) = \frac{\epsilon' + \cosh\dfrac{u}{1-\sigma}}{1+\epsilon'} \qquad (41)$$

and

$$\chi = \theta_2 - \alpha' - \tfrac{1}{2}\beta'. \qquad (42)$$

Equating real parts of each side of (40) gives

$$f(u) = \frac{1+\sec\beta'+\tan\beta'\sin\chi}{1+\cos\chi}, \qquad (43)$$

and this yields the following equation:

$$\tan^2\tfrac{1}{2}\chi + 2\tan\tfrac{1}{2}\beta'\tan\tfrac{1}{2}\chi + 1 = \frac{2\cos\beta}{1+\cos\beta}f(u), \qquad (44)$$

from which χ may be determined immediately.

Hence, for any point z_2 on the aerofoil corresponding to $u + i\sigma\pi$, θ_2 may readily be evaluated. The reverse process is equally convenient.

10. *Relationship to Joukowski's Transformation.*

The present system can be interpreted in an interesting way with reference to the simple transformation originally proposed by Joukowski. In order to obtain by the present method an aerofoil profile having a cusp at its trailing edge, as characterizes Joukowski aerofoil sections, it is found necessary to invert a parabola instead of a hyperbola. The sequence of transformations are then modified to

$$z_1 = \epsilon + i\delta + w^2,$$
$$t = \epsilon' + i\delta' + (w - iv_0)^2,$$
$$t(t_1 + 2) = 4\epsilon' \sec \beta' e^{i\beta'},$$

and (22).

It is easily proved that these are equivalent to the transformation

$$z_2 = t_2 + 1/t_2$$

applied to the circle

$$\mid t_2 + i \tan \tfrac{1}{2}\beta' \mid \; = \sec \tfrac{1}{2}\beta'.$$

That the Joukowski aerofoil is the inverse of a parabola does not appear to be generally recognized, although it is apparently known in Russia by that name. However, the graphical description of a Joukowski aerofoil by Trefftz's well known construction is already a very simple matter.

For reasons already discussed, cusped profiles are of restricted interest, and in the following sections we return to the generally useful shapes now obtained.

11. *The Irrotational Flow past the Aerofoil.*

In the general case the potential function W of the flow past the circle is given by

$$\mathrm{W} = \phi + i\psi = -\mathrm{U}' \left(t_2 + \frac{a^2}{t_2} \right) - \frac{i\mathrm{K}}{2\pi} \log \frac{t_2}{a}, \quad . \quad . \quad (45)$$

where U' is the undisturbed velocity and K the circulation. The complex velocity is

$$\frac{d\mathrm{W}}{dt_2} = -\mathrm{U}' \left(1 - \frac{a^2}{t_2{}^2} \right) - \frac{i\mathrm{K}}{2\pi t_2} . \quad . \quad . \quad (46)$$

According to Joukowski's hypothesis, the flow in the z_2-plane will leave the trailing edge of the aerofoil smoothly if B (fig. 7 (e)) is arranged to be a stagnation point in the flow past the circle, *i. e.*, if

$$K = 4\pi a U' \sin (\alpha' + \tfrac{1}{2}\beta'). \quad . \quad . \quad . \quad (47)$$

This determination of K requires an empirical factor which, however, is for experiment to establish and need not be discussed here.

On the circular boundary we have

$$\phi = -2U' [\cos \theta_2 - \theta_2 \sin (\alpha' + \tfrac{1}{2}\beta')], \quad . \quad . \quad (48)$$

and for the "velocity of the transformation"

$$h = 2U'[\sin \theta_2 + \sin (\alpha' + \tfrac{1}{2}\beta')]. \quad . \quad . \quad (49)$$

It follows from (35) that the angle of zero lift of the aerofoil given when $\alpha' = -\tfrac{1}{2}\beta'$, is

$$\tfrac{1}{2}\beta' - \beta + \arg \left(\frac{\sinh w_c}{\sinh p_c} \right).$$

The complex velocity at any point on the aerofoil boundary follows from the evaluation of

$$h \left| \frac{dt_2}{dw} \right| . \left| \frac{dw}{dz_2} \right|. \quad . \quad . \quad . \quad . \quad (50)$$

From (39) we obtain from differentiation, after putting in boundary values and reducing,

$$\frac{\left| \sinh \dfrac{u}{1-\sigma} \right|}{\left| \lambda \right| (1-\sigma)} . \left| \frac{dw}{dt_2} \right| = \frac{\cos^2 \tfrac{1}{2}\beta' . \left| \sin \tfrac{1}{2}(\chi + \beta') \right|}{4 \left| \cos^3 \tfrac{1}{2}\chi \right|}. \quad (51)$$

Hence the complex velocity at the aerofoil boundary is finally equal to

$$\frac{\left| h \right|}{x_2{}^2 + y_2{}^2} . \left| \frac{\cos \beta' . \cos^3 \tfrac{1}{2}\chi}{\cos^2 \tfrac{1}{2}\beta' . \sin \tfrac{1}{2}(\chi + \beta')} \right| . \left| \frac{\sinh \dfrac{u}{1-\sigma}}{\sinh (u + i\sigma\pi)} \right|$$

$$. \frac{1}{(1-\sigma) \left| 1 - \cosh p_c \right|}. \quad . \quad . \quad (52)$$

The pressure distribution round this boundary follows immediately by Bernoulli's Theorem.

12. Pitching Moment.

The lift of the aerofoil is, of course, ρUK per unit length, ρ being the density of the fluid. But in aeronautical applications it is usually required also to be able to calculate the moment of this force about some point in the plane of the section and, further, to determine how this " pitching " moment varies when the angle of incidence α of the aerofoil is changed.

To calculate the moment about the origin in the z_2-plane we have to evaluate

$$M = \tfrac{1}{2}\rho \ \mathrm{R.P.} \oint z_2 \left(\frac{dW}{dz_2}\right)^2 dz_2, \quad . \quad . \quad . \quad (53)$$

where the integral is taken round a contour enclosing the profile. As before, W is the complex potential (on this contour).

Now

$$z_2 \left(\frac{dW}{dz_2}\right)^2 dz_2 = z_2 \left(\frac{dW}{dt_2}\right)^2 \frac{dt_2}{dz_2} \, dt_2, . \quad . \quad . \quad (54)$$

and we have

$$\left(\frac{dW}{dt_{2'}}\right)^2 = \left(U' + \frac{iK}{2\pi t_2} - \frac{a^2 U'}{t_2{}^2}\right)^2$$

$$= U'^2 + \frac{iKU'}{\pi t_2} - \left(\frac{K^2}{4\pi^2} + 2a^2 U'^2\right)\frac{1}{t_2{}^2} + \ldots \quad (55)$$

or, say,

$$= A_0 + A_1/t_2 + A_2/t_2{}^2 + \ldots \quad . \quad . \quad . \quad (56)$$

Also

$$z_2 = a_0 t_2 + a_1 + a_2/t_2 + \ldots, \quad . \quad . \quad . \quad (57)$$

where the values of the coefficients are known.

Hence

$$z_2 \left(\frac{dW}{dt_2}\right)^2 \frac{dt_2}{dz_2} = \left(A_0 + \frac{A_1}{t_2} + \frac{A_2}{t_2{}^2} + \ldots\right)$$

$$\times \left(1 + \frac{a_1}{a_0 t_2} + \frac{a_2}{a_0 t_2{}^2} + \ldots\right)\left(1 + \frac{a_2}{a_0 t_2{}^2} + \ldots\right) t_2.$$

The coefficient of $1/t_2$ in this continued product is

$$2A_0 \frac{a_2}{a_0} + A_1 \frac{a_1}{a_0} + A_2.$$

Hence, finally,

$$M = \rho\pi I.P. \left[2U'^2 \frac{a_2}{a_0} + \frac{iKU'}{\pi} \frac{a_1}{a_0} \right], \quad . \quad . \quad (58)$$

A_2 being wholly real.

When the aerofoil is at its angle of zero lift, the second term in this expression vanishes. In these circumstances we have, for the zero lift-moment M_0,

$$M_0 = \rho\pi I.P.[2U'^2 . a_2/a_0]. \quad . \quad . \quad . \quad (59)$$

To evaluate, we first express the a-coefficients in terms of b-coefficients by (31), obtaining

$$\frac{a_2}{a_0} = \frac{b_2 - b_1 b_3}{b_1^2}.$$

Then from (30) this becomes

$$\frac{a_2}{a_0} = e^{2i\alpha'} \left[1 + \lambda^2 \frac{c_2^2 - c_1 c_3}{c_1^2} \right].$$

Finally, substituting for the c-cofficients, and after a little reduction, we find

$$\frac{a_2}{a_0} = e^{2i\alpha'} \left\{ 1 + \frac{\lambda^2}{4 \sinh^4 p_c} \left[(1-\sigma)^2 \sinh^2 p_c \left(\frac{1}{3} + \frac{1}{\sinh^2 w_c} \right) \right. \right.$$
$$\left. \left. - \sinh^2 p_c \left(\frac{1}{3} + \frac{1}{\sinh^2 p_c} \right) \right] \right\}$$
$$= e^{2i\alpha'} \left\{ 1 + \frac{4R_2^2}{R_1^2} e^{2i(\psi_2 - \psi_1)} \left[(1-\sigma)^2 \left(\frac{1}{3} + \frac{e^{-2i\psi_3}}{R_3^2} \right) \right. \right.$$
$$\left. \left. - \left(\frac{1}{3} + \frac{e^{-2i\psi_2}}{R_2^2} \right) \right] \right\},$$

where

$$R_1^2 = \left(\cosh \frac{u_c}{1-\sigma} + \cos \frac{v_c - \sigma\pi}{1-\sigma} \right)^2,$$

$$\tan\psi_1 = \frac{\sinh \dfrac{u_c}{1-\sigma} \sin \dfrac{v_c - \sigma\pi}{1-\sigma}}{1 + \cosh \dfrac{u_c}{1-\sigma} \cos \dfrac{v_c - \sigma\pi}{1-\sigma}}.$$

$$R_2^2 = \cosh^2 \frac{u_c}{1-\sigma} - \cos^2 \frac{v_c - \sigma\pi}{1-\sigma},$$

$$\tan\psi_2 = \coth \frac{u_c}{1-\sigma} \tan \frac{v_c - \sigma\pi}{1-\sigma}.$$

$$R_3^2 = \cosh^2 u_c - \cos^2 v_c, \quad \tan\psi_3 = \coth u_c \tan v_c,$$

and from (26)

$$\beta' = \arg\,(1-\cosh\,p_c) = 2\psi_2 - \psi_1 - \pi.$$

Hence

$$M_0 = 2\rho U'^2 \pi \left[-\sin\beta' + \frac{4R_2^2}{R_1^2} \left\{ \frac{(1-\sigma)^2-1}{3} \sin\psi_1 \right. \right.$$
$$\left. \left. + \frac{(1-\sigma)^2}{R_3^2} \sin\,(\psi_2+2\psi_3) + \frac{1}{R_2^2} \sin\,(2\psi_1+\beta') \right\} \right],$$
$$\qquad\qquad\qquad . \quad . \quad . \quad (60)$$

a result which depends upon both camber and tail angle.

It is convenient and usual to work with a moment coefficient :

$$k_{m0} = M_0/\rho U^2 c^2.$$

In this formula, c, the chord of the aerofoil, is given by (8). U, the velocity at infinity in the aerofoil plane, is conveniently re-expressed from (3) in the form

$$U = 4U'(1-\sigma)(1+\epsilon')\sec\beta'.\, R_3/R_2.$$

Hence the expression for the moment coefficient finally becomes

$$k_{m0} = \frac{\pi}{8}\,.\,\mathrm{I.\,P.}\,\left(\frac{a_2}{a_0}\right)\frac{(1+\cos\,\sigma\pi)^2 R_2^2\cos^2\beta'}{(1+\epsilon')^2(1-\sigma)^2\,R_3^2\cos^2\beta}\,.\quad(61)$$

Fig. 8 has been prepared to compare the pitching moment for the present series of aerofoils with those for the simple Joukowski and Kármán-Trefftz types. Curve (a) gives the variation of k_{m0} with tail angle for the new family. Curve (b) applies to Kármán-Trefftz aerofoils having approximately the same mean cambers and angles of zero lift. The tail angle of simple Joukowski aerofoils is zero, but the dotted curve (c) shows k_{m0} for a series of this type having approximately the same mean cambers and angles of zero lift as the aerofoils of curves (a) and (b). With aeroplane wings it is desirable that the pitching moment M_0 should be as small as possible in order to economize in structural weight for a given strength. The present series of shapes is seen to have an advantage in this respect. The advantage was anticipated, and also the fact that it is small in magnitude. For the pitching moment M_0 to vanish it is known that the profile must be reflexed towards the trailing edge. Although this modification is not always convenient to reproduce in aeroplane wings, there appears no reason why it should not be incorporated in the present series

of shapes. The development remains, however, for further investigation.

Fig. 8.
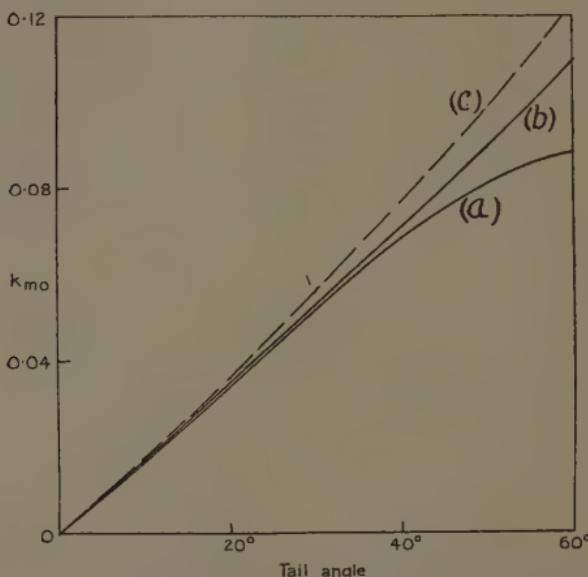
(a) The new profiles. (b) Kármán-Trefftz series. (c) Simple Joukowski shapes of equal camber and zero lift incidence.

13. *Conclusion.*

A new family of symmetrical and cambered streamline sections have been mapped from the circle by a sequence of conformal transformations. The shapes obtained are suitable for the sections of wings, struts, fins, and airscrew blades ; judged by modern aeronautical practice they appear, indeed, to approach perfection in this respect, and to represent a distinct advance on earlier transformed profiles of slender form. They also generate a reduced pitching moment. The new profiles are convenient to handle analytically, and may be traced by simple mechanical means.

It appears probable that a modification can be introduced to yield a stationary centre of pressure, if desired, without losing the benefit of conformal transformation. This possible development is under investigation.

XLIII. *Notes on Two-Dimensional Potential Theory.—*
V. *The Generalized Formulæ for the Forces and Couple
on a Moving Cylinder.* By Rosa M. Morris, *B.Sc.**

1. THERE have been up to the present two methods
of finding the forces and couple on a cylinder in
motion in a fluid. The first is the Lagrangian method,
whereby the forces can be found once the expression
for the kinetic energy has been obtained. This method
is restricted in so far that it at once leads to difficulties
when circulation is introduced, since the energy of any
simple two-dimensional circulation is always infinite.
The second method is to calculate the forces indirectly
by Kirchhoff's method, employing the idea of the
"impulse," but so far it has only been possible to find
the forces by this method by separating the complex
potential into its real and imaginary parts, and then only
for but very few cases. Glauert †, and later Lamb ‡,
in papers issued by the Aeronautical Research Committee,
calculate the forces in this way in terms of the real
velocity potential of the motion, and both work out the
well-known case of the elliptic cylinder. Bickley §,
at about the same time, and using the same method,
succeeds in finding the force and couple on a general
cylinder in uniform motion, including rotation, applying
his results in particular to Joukowski aerofoils; but he
does not take into account the accelerations of the
motion in the general case.

The object of this paper is to show that there are
generalized forms of the simple Blasius formulæ applicable
to the case of a cylinder in motion in any manner in a
fluid, and which enables the forces and couple to be
evaluated with much greater ease than by any other
method. The formulæ involve the complex potential
function and the variables defining the motion of the
fluid and cylinder.

2. The cylinder is assumed to be moving perpendicular
to its axis with velocity components u, v along axes

Ox, Oy on a normal cross-section of the cylinder, and has an angular velocity ω about any normal to this cross-section (parallel to the axis). Supposing the cylinder to be defined in the usual way by

$$z = z(\zeta) \; ; \quad \eta = 0,$$

where $\zeta = \xi + i\eta$, the complex velocity potential Ω for the motion is found by the usual method expounded at some length in a previous note. To calculate the forces we want the normal pressure p at the surface of the cylinder. If we know this then the components X, Y of the resultant force on the cylinder are

$$\mathrm{X} = -\int_c p \, dy, \qquad \mathrm{Y} = \int_c p \, dy \, ;$$

so that obviously

$$\mathrm{Y} + i\mathrm{X} = \int_c p \, d\overline{z},$$

integrated round c, the curve of cross-section of the cylinder. The pressure at any point in the fluid is given by the usual pressure equation

$$p = p_0 - \rho \frac{\partial \phi}{\partial t} - \tfrac{1}{2}\rho q^2,$$

where ϕ is the real velocity potential, chosen so that its positive gradients determine the velocities, and q is the velocity of the fluid at the point. Since the complex potential function is defined by

$$\Omega = \phi + i\psi, \qquad \overline{\Omega} = \phi - i\psi,$$

we have

$$\frac{\partial \Omega}{\partial z} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x}; \qquad \frac{\partial \overline{\Omega}}{\partial x} = \frac{\partial \phi}{\partial x} - i \frac{\partial \psi}{\partial x},$$

so that

$$q^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2 = \frac{\partial \Omega}{\partial z} \, \frac{\partial \overline{\Omega}}{\partial z},$$

and the pressure equation can now be expressed in terms of Ω in the form

$$p = p_0 - \tfrac{1}{2}\rho \left\{ \frac{\partial \Omega}{\partial t} + \frac{\partial \overline{\Omega}}{\partial t} + \frac{\partial \Omega}{\partial z} \, \frac{\partial \overline{\Omega}}{\partial z} \right\}.$$

If now there are no accelerations the conditions relative to the moving axes must be steady, so that

$$- \frac{\partial \phi}{\partial t} - u \frac{\partial \phi}{\partial x} - v \frac{\partial \phi}{\partial y} + \omega \left(y \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial y} \right) = 0,$$

with a similar equation for ψ. Multiplying the ψ equation by i, and adding, and remembering that

$$\frac{\partial \Omega}{\partial z} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{1}{i} \left(\frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} \right),$$

we have

$$\frac{\partial \Omega}{\partial t} = -(w + i\omega z) \frac{\partial \Omega}{\partial z},$$

where

$$w = u + iv,$$

and then

$$\frac{\partial \bar{\Omega}}{\partial t} = -(\bar{w} - i\omega \bar{z}) \frac{\partial \bar{\Omega}}{\partial z}.$$

Now Ω being a function of the velocity of the cylinder, and also such things as circulation, $\dfrac{\partial \Omega}{\partial t}$ will, in addition to the above, contain terms involving the accelerations and the variation in the circulation. Taking first only those terms which arise when these accelerations are zero, they contribute to p the amount

$$\tfrac{1}{2}\rho \left[(w + i\omega z) \frac{\partial \Omega}{\partial z} + (\bar{w} - i\omega z) \frac{\partial \bar{\Omega}}{\partial z} - \frac{\partial \Omega}{\partial z} \frac{\partial \bar{\Omega}}{\partial z} \right],$$

which is the same as

$$\tfrac{1}{2}\rho \left[(w + i\omega z)(w - i\omega z) - \left(w + i\omega z - \frac{\partial \bar{\Omega}}{\partial z} \right) \left(\bar{w} - i\omega z - \frac{\partial \Omega}{\partial z} \right) \right],$$

so that the corresponding contribution to the force components X, Y is given by

$$2(\mathrm{Y} + i\mathrm{X})/\rho = \int_c (w + i\omega z)(\bar{w} - i\omega \bar{z}) \, d\bar{z}$$

$$- \int_c \left(w + i\omega z - \frac{\partial \bar{\Omega}}{\partial z} \right) \left(\bar{w} - i\omega \bar{z} - \frac{\partial \Omega}{\partial z} \right) dz.$$

But the relative motion at the boundary is tangential, and therefore

$$\frac{u-\omega y-\dfrac{\partial\phi}{\partial x}}{dx}=\frac{v+\omega x-\dfrac{\partial\phi}{\partial y}}{dy}=\frac{w+i\omega z-\dfrac{\overline{\partial\Omega}}{\partial z}}{dz}=\frac{\overline{w}-i\omega\dot{z}-\dfrac{\partial\Omega}{\partial z}}{d\bar{z}};$$

so that we have, apart from the acceleration terms,

$$2(\mathrm{Y}+i\mathrm{X})/\rho=\int_c (w+i\omega z)(\overline{w}-i\omega z)\,d\bar{z}.$$
$$-\int_c \left(\overline{w}-i\omega z-\frac{\partial\Omega}{\partial z}\right)^2 dz.$$

In a similar manner we see that the couple Γ is given by

$$\Gamma=\int_c p(x\,dx+y\,dy),$$

which is the real part of

$$\int_c pz\,d\bar{z}.$$

Arguing in the same manner as above we find that the couple, apart from acceleration terms, is the real part of

$$\tfrac{1}{2}\rho\int_c (w+i\omega z)(w-i\omega\tilde{z})z\,dz-\tfrac{1}{2}\rho\int_c \left(\overline{w}-i\omega z-\frac{\partial\Omega}{\partial z}\right)^2 z\,dz.$$

These two integrals appear to be the general forms of the familiar statical results due to Blasius. They have, of course, to be generalized when there are accelerations by the addition of terms in the accelerations. These are, in the formula for $\mathrm{Y}+i\mathrm{X}$,

$$-\tfrac{1}{2}\rho\int_c \left\{ \frac{\partial\Omega}{\partial w}\dot{w}+\frac{\overline{\partial\Omega}}{\partial w}\dot{\overline{w}}+\frac{\partial\Omega}{\partial\overline{w}}\dot{\overline{w}}+\frac{\overline{\partial\Omega}}{\partial\overline{w}}\dot{w}+\frac{\partial\Omega}{\partial\omega}\dot{\omega} \right.$$
$$\left. +\frac{\overline{\partial\Omega}}{\partial\omega}\dot{\omega}+\frac{\partial\Omega}{\partial\mathrm{K}}\dot{\mathrm{K}}+\frac{\overline{\partial\Omega}}{\partial\mathrm{K}}\dot{\overline{\mathrm{K}}} \right\}d\bar{z},$$

and in the formula for the couple

$$-\tfrac{1}{2}\rho\int_c \left\{ \frac{\partial\Omega}{\partial w}\dot{w}+\frac{\overline{\partial\Omega}}{\partial w}\dot{\overline{w}}+\frac{\partial\Omega}{\partial\overline{w}}\dot{\overline{w}}+\frac{\overline{\partial\Omega}}{\partial\overline{w}}\dot{w}+\frac{\partial\Omega}{\partial\omega}\dot{\omega} \right.$$
$$\left. +\frac{\partial\Omega}{\partial\omega}\dot{\omega}+\frac{\partial\Omega}{\partial\mathrm{K}}\dot{\mathrm{K}}+\frac{\overline{\partial\Omega}}{\partial\overline{\mathrm{K}}}\dot{\overline{\mathrm{K}}} \right\}z\,d\bar{z}.$$

3. To illustrate the use of these formulæ let us consider the case of the elliptic cylinder moving in the manner described where u, v, and ω are variable, and also include in the motion a variable circulation of strength K round the cylinder.

The elliptic cylinder is defined by the transformation

$$z = c \cos (\zeta + i\alpha) \; ; \quad \eta = 0,$$

and by the usual method we find the complex potential for the above motion to be given by

$$\Omega = \tfrac{1}{2} c e^{i\zeta} \{ \bar{w} e^{-\alpha} - w e^{\alpha} \} - \tfrac{1}{2} i \omega c^2 e^{2i\zeta} + \frac{\mathrm{K} \zeta}{2\pi}.$$

Taking first the acceleration terms : these contribute to p the amount

$$-\tfrac{1}{2} \rho \left\{ \; \frac{\partial \Omega}{\partial w} \dot{w} + \frac{\overline{\partial \Omega}}{\partial w} \dot{\bar{w}} + \frac{\partial \Omega}{\partial w} \dot{\bar{w}} + \frac{\overline{\partial \Omega}}{\partial w} \dot{w} + \frac{\partial \Omega}{\partial \omega} \dot{\omega} \right.$$
$$\left. + \frac{\overline{\partial \Omega}}{\partial \omega} \dot{\omega} + \frac{\partial \Omega}{\partial \mathrm{K}} \dot{\mathrm{K}} + \frac{\overline{\partial \Omega}}{\partial \mathrm{K}} \dot{\mathrm{K}} \; \right\},$$

which give in this case

$$\tfrac{1}{2} \rho \left\{ \; c \dot{w} i \sin (\zeta - i\alpha) - c \dot{\bar{w}} i \sin (\zeta + i\alpha) - \tfrac{1}{2} \dot{\omega} c^2 \sin 2\zeta - \frac{\mathrm{K} \zeta}{\pi} \right\}.$$

The contribution of these terms to the components of the force will therefore be

$$-\tfrac{1}{2} \rho c \int_0^{2\pi} \left\{ \; c \dot{w} i \sin (\zeta - i\alpha) - c \dot{\bar{w}} i \sin (\zeta + i\alpha) \right.$$
$$\left. - \tfrac{1}{2} \dot{\omega} c^2 \sin 2\zeta - \frac{\dot{\mathrm{K}} \zeta}{\pi} \right\} \sin (\zeta - i\alpha) \, d\zeta,$$

which gives

$$-\tfrac{1}{2} \pi \rho c \left\{ \; c i \dot{w} - c i \dot{\bar{w}} \cosh 2\alpha - \frac{2 \dot{\mathrm{K}}}{\pi} \cosh \alpha \; \right\}.$$

The terms which arise when the accelerations are zero contribute to the components of the force the amount

$$\tfrac{1}{2} \rho \int_c (w + i\omega z)(\bar{w} - i\omega z) \, dz - \tfrac{1}{2} \rho \int \left(\bar{w} - i\omega z - \frac{\partial \Omega}{\partial z} \right)^2 dz.$$

The integrals involved in this are

$$\int_c dz = 0; \quad \int_c d\bar{z} = 0; \quad \int_c \bar{z}\, d\bar{z} = 0;$$

$$\int_c z\bar{z}\, d\bar{z} = -c^3 \int_0^{2\pi} \cos(\zeta + i\alpha) \cos(\zeta - i\alpha) \sin(\zeta - i\alpha)\, d\zeta = 0;$$

$$\int_c \bar{z}^2\, dz = -c^3 \int_0^{2\pi} \cos^2(\zeta - i\alpha) \sin(\zeta + i\alpha)\, d\zeta = 0;$$

also

$$\int z\, d\bar{z} = -c^2 \int \cos(\zeta + i\alpha) \sin(\zeta - i\alpha)\, d\zeta$$
$$= \pi i c^2 \sinh 2\alpha,$$

and

$$\int_c (z\, d\bar{z} + \bar{z}\, dz) = \int_c d(z\bar{z}) = 0;$$

so that

$$\int \bar{z}\, dz = -\pi i c^2 \sinh 2\alpha.$$

Then

$$\int_c \frac{\partial \Omega}{\partial z}\, dz = \mathbf{K}$$

and

$$\int_c \bar{z}\, \frac{\partial \Omega}{\partial z}\, dz = \int_0^{2\pi} \bar{z}\, \frac{\partial \Omega}{\partial \zeta}\, d\zeta,$$

which is

$$c \int_0^{2\pi} \cos(\zeta - i\alpha) \left\{ \tfrac{1}{2} i c e^{i\zeta}(\bar{w} e^{-\alpha} - w e^{\alpha}) + \tfrac{1}{2}\omega c^2 e^{2i\zeta} + \frac{\mathbf{K}}{2\pi} \right\} d\zeta$$

$$= \frac{c}{2} \int_0^{2\pi} (e^{i\zeta + \alpha} + e^{-i\zeta - \alpha}) \left\{ \tfrac{1}{2} i c e^{i\zeta}(\bar{w} e^{-\alpha} - w e^{\alpha}) \right.$$
$$\left. + \tfrac{1}{2}\omega c^2 e^{2i\zeta} + \frac{\mathbf{K}}{2\pi} \right\} d\zeta;$$

and since all terms of the form $e^{ni\zeta}$ in the integrand give zero, except those for which $n = 0$, this gives

$$\tfrac{1}{2}\pi i c^2 (\bar{w} e^{-2\alpha} - w).$$

Further, we have

$$\int_c \left(\frac{\partial \Omega}{\partial z} \right)^2 dz = \int_0^{2\pi} \left(\frac{\partial \Omega}{\partial \zeta} \right)^2 \frac{d\zeta}{dz/d\zeta}$$

$$= -\int_0^{2\pi} \left\{ \tfrac{1}{2} ice^{i\zeta}(\bar{w}e^{-\alpha} - we^{\alpha}) + \tfrac{1}{2}\omega c^2 e^{2i\zeta} + \frac{\mathrm{K}}{2\pi} \right\} \frac{d\zeta}{c \sin (\zeta + i\alpha)},$$

and by expanding the denominator in powers of $e^{i\zeta}$ this integral is easily seen to be zero.

Thus in all we have

$$\tfrac{1}{2}\rho[\pi c^2 \omega \bar{w} \sinh 2\alpha + \pi c^2 \omega(\bar{w}e^{-2\alpha} - w) + 2\bar{w}\mathrm{K}],$$

and, finally, including the acceleration terms, the components of the force are given by

$$2(\mathrm{Y} + i\mathrm{X})/\rho = -\pi i c^2 \dot{w} + \pi i c^2 \dot{\bar{w}} \cosh 2\alpha + 2c\dot{\mathrm{K}} \cosh \alpha$$
$$+ \pi c^2 \omega \bar{w} \cosh 2\alpha - c^2 \omega w + 2\bar{w}\mathrm{K}.$$

The acceleration terms in the couple are in a similar way given by the real part of

$$-\tfrac{1}{2}\rho c^2 \int_0^{2\pi} \left\{ ci\dot{w} \sin (\zeta - i\alpha) - ci\dot{\bar{w}} \sin (\zeta + i\alpha) \right.$$
$$\left. -\tfrac{1}{2}\dot{\omega}c^2 \sin 2\zeta - \frac{\dot{\mathrm{K}}\zeta}{\pi} \right\} \cos (\zeta + i\alpha) \sin (\zeta - i\alpha) \, d\zeta$$
$$= -\tfrac{1}{4}\rho c^2 \dot{\mathrm{K}}(1 + 2\pi i \sinh 2\alpha) + \tfrac{1}{8}\rho c^4 \dot{\omega},$$

the real part of which is

$$-\tfrac{1}{4}\rho c^2 \dot{\mathrm{K}} + \tfrac{1}{8}\pi \rho c^4 \dot{\omega}.$$

The steady motion terms are the real part of

$$\tfrac{1}{2}\rho \int_c (w + i\omega z)(\bar{w} - i\omega\bar{z})z \, d\bar{z} - \tfrac{1}{2}\rho \int_c \left(\bar{w} - i\omega\bar{z} - \frac{\partial \Omega}{\partial z} \right)^2 z \, dz,$$

and the extra integrals involved are

$$\int_c (z^2\bar{z} \, dz + \bar{z}^2 z \, dz) = \int z\bar{z} \, d(z\bar{z}) = |zz|^2 = 0 \; ;$$

$$\int z^2 \, d\bar{z} = -c^3 \int \cos^2 (\zeta + i\alpha) \sin (\zeta - i\alpha) \, d\zeta = 0 \; ;$$

$$\int_c z\left(\frac{\partial\Omega}{\partial z}\right)^2 dz = \int_0^{2\pi} z\left(\frac{\partial\Omega}{\partial\zeta}\right)^2 \frac{d\zeta}{dz/d\zeta}$$

$$= -\int_0^{2\pi} \cos(\zeta+i\alpha)\left\{\begin{array}{l}\tfrac{1}{2}ice^{i\zeta}(\overline{w}e^{-\alpha}-we^{\alpha}) \\[4pt] \qquad +\tfrac{1}{2}\omega c^2 e^{2i\zeta}+\dfrac{K}{2\pi}\end{array}\right\}\frac{d\zeta}{\sin(\zeta+i\alpha)},$$

which by expanding the denominator is again seen to be zero.

Further, we have,

$$\int_c z\,\frac{\partial\Omega}{\partial z}dz = \int_0^{2\pi} z\left(\frac{\partial\Omega}{\partial\zeta}\right)d\zeta$$

$$= c\int_0^{2\pi}\cos(\zeta+i\alpha)\left\{\begin{array}{l}\tfrac{1}{2}ice^{i\zeta}(\overline{w}e^{-\alpha}-we^{\alpha}) \\[4pt] \qquad +\tfrac{1}{2}\omega c^2 e^{2i\zeta}+\dfrac{K}{2\pi}\end{array}\right\}d\zeta,$$

which in a similar manner as before gives

$$\tfrac{1}{2}\pi i c^2(\overline{w}-we^{2\alpha}).$$

Also

$$\int_c z\bar{z}\,\frac{\partial\Omega}{\partial z}\,dz$$

$$= c^2\int_0^{2\pi}\cos(\zeta+i\alpha)\cos(\zeta-i\alpha)\left\{\begin{array}{l}\tfrac{1}{2}ice^{i\zeta}(\overline{w}e^{-\alpha}-we^{\alpha}) \\[4pt] \qquad +\tfrac{1}{2}\omega c^2 e^{2i\zeta}+\dfrac{K}{2\pi}\end{array}\right\}d\zeta$$

$$= \tfrac{1}{2}c^2\int_0^{2\pi}(\cos 2\zeta-\cosh 2\alpha)\,\frac{\partial\Omega}{\partial\zeta}\,d\zeta$$

$$= \pi c^2\left\{\tfrac{1}{4}\omega c^2-\frac{K}{2\pi}\cosh 2\alpha\right\}.$$

Thus in all we have the couple as the real part of

$$-\tfrac{1}{4}\rho c^2\dot{K}+\tfrac{1}{8}\pi\rho c^4\dot{\omega}-\tfrac{1}{2}\pi i\rho c^2 w\overline{w}\cosh 2\alpha$$
$$+\tfrac{1}{2}\pi i\rho c^2\overline{w}^2-\tfrac{1}{4}\pi i\rho c^4\omega^2+\tfrac{1}{2}i\rho c^2 K\omega\cosh 2\alpha,$$

or $$\varGamma = -\tfrac{1}{4}\rho c^2\dot{K}+\tfrac{1}{8}\pi\rho c^4\dot{\omega}+\pi\rho c^2 uv,$$

forms agreeing with those obtained by Glauert.

The only terms contributing to the couple are due to acceleration in the angular velocity and the rate of change of the circulation, and also to the skewness of the linear motion, the latter term corresponding to the usual statical couple on the cylinder in the uniform stream.

University College, Cardiff.
April 27th, 1937.

XLIV. *The End-correction of an Unflanged Pipe. By* A. E. BATE, *Ph.D., M.Sc., F.Inst.P., Northern Poly-technic, London* *.

ABSTRACT.

THE conductivity of a conducting disk embedded in an infinite medium of specific conductivity k is equal to 8 kr, r being the radius of the disk. This is shown to become $4kr\sqrt{2}$ when one face is insulated, and when applied to the question of the effect of the removal of a flange on the end-correction of a flanged pipe it is shown that the correction is diminished to $(0\cdot5)^{\frac{1}{2}}$ of its former value, *i. e.*, from $0\cdot82\,r$ to $0\cdot58\,r$.

§ 1. THE end-correction of a pipe of radius r ter- minating in an infinite flange was shown by the late Lord Rayleigh to lie between $(\pi/4)\,r$ and $(8/3\pi)\,r$. He gave $0\cdot82\,r$ as probably the true value [1], and this has recently been confirmed theoretically by L. V. King [2].

The correction for the pipe without the flange has not yet been derived theoretically, but it is generally accepted [3] as being $0\cdot58\,r$. It is the purpose of this paper to justify this on theoretical grounds.

§ 2. In determining the lower limit quoted above Rayleigh [4] assumed the vibrations outside the flanged pipe to be set up by a disk at the end of the pipe and of equal radius. The conductivity † of the orifice is there- for $4\,r$.

This value was obtained by Weber [5] in calculating the resistance to the flow of a current of electricity from a charged disk of the same radius attached to the surface of a semi-infinite medium of unit conductivity. Weber also showed that if the disk be embedded in an infinite medium the resistance is halved, the conductivity becoming $8r$ *. This result is obtained if the infinite medium be supposed divided by an insulating plane lamina of infinite extent in the plane of the disk, for the two sides, each of conductivity $4r$, would act in parallel. Again, if a half disk be arranged with the diameter in the surface of the semi-infinite medium and normal to it, the conductivity is also $4r$.

§ 3. The present problem is the determination of the effect due to the removal of the flange from a pipe on the conductivity of the end of the pipe. As the energy from the unflanged pipe flows into an infinite atmosphere, but at first in one direction only, the divergence is *not* completely spherical, and the problem resembles that of the disk embedded in the infinite medium, but with one face insulated. In this case the conductivity must lie between that of the disk in the surface of the semi-infinite medium and that due to the disk in the infinite medium, *i. e.*, between $4r$ and $8r$.

§ 4. Two disks of equal radius situated in the surface of a semi-infinite conductor of unit conductivity have the same conductivity if they are separated by a great distance, and this becomes $8r$ when they are acting in parallel. When, however, they are close together there is mutual interference which reduces the combined conductivity. If, finally, they merge to form a single disk of double area [6], [7] the total conductivity becomes $4(r\sqrt{2})$, as the radius of the new disk is $r\sqrt{2}$.

* Crandall points out that the only impedance offered by the moving mass of air in a *tuned* resonator opening is its radiation resistance (p. 175), and, further, that the radiation resistance for the unflanged end of a tube is, for low frequencies, half as great for *complete* spherical divergence as for divergence into a semi-infinite medium *if the end of the tube be regarded as an isolated point source* (p. 151) (italics mine) : ' Vibrating Systems and Sound.'

Stenzel has shown that, for an open *flanged* pipe giving a frequency whose wave-length is greater than about $10r$ the end of the pipe may be regarded as a point source, *i. e.*, the divergence is spherical into the *semi*-infinite medium [3a] : Reference (3 *b*), p. 216.

Thus the interference between the effluxes from the two sources of equal conductivity which arises when they unite reduces the total conductivity to $1 \surd 2$ of their sum.

§ 5. Turning again to the completely embedded disk, it may be divided into two equal parts by an infinite

Fig. 1.
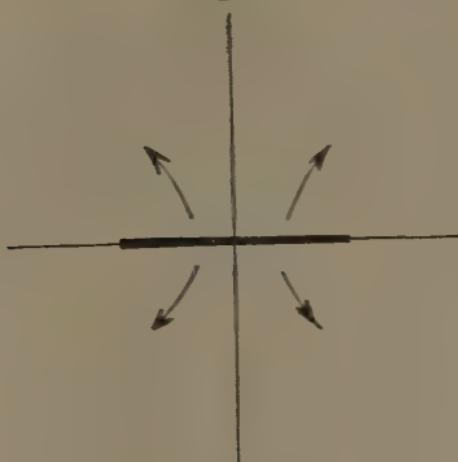
Fig. 2.

plane insulating lamina normal to its plane, and, again, by a similar plane in its own plane (fig. 1). The infinite material is now symmetrically divided into four equal portions, each with a semicircular conductor. The conductivity into each quarter-infinite atmosphere is clearly $2r$, as the presence of the planes does not affect the flow. If all semicircular disks but one be removed (fig. 2) the lines of flow in the untouched quadrant will

remain as before, owing to the presence of the planes.
Removal of the horizontal plane will allow efflux into the
fourth quadrant (fig. 3), but this is prevented if the
semicircular disk in that quadrant is active.

Alternatively, if the perpendicular plane is removed
from the half disk of fig. 2, leaving the horizontal plane

Fig. 3.
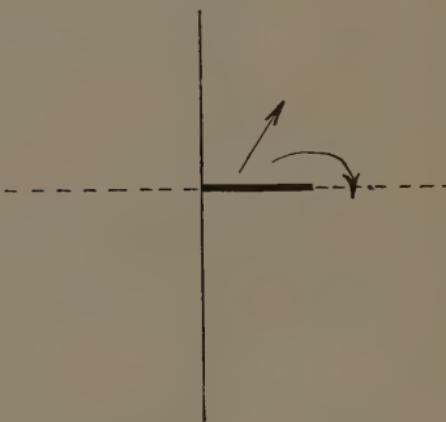
Fig. 4.
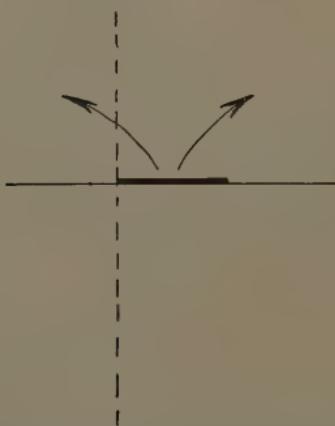
(fig. 4), efflux into the second quadrant will occur ; if the
half disk in that quadrant is active, however, the effluxes
will be constrained as by the plane (fig. 5) to their
respective quadrants.

As the complete conductivity into the first and second
quadrants (fig. 1) equals that of the complete con-
ductivity into the first and fourth (see § 2), it is clear

that the effect of the efflux in the fourth quadrant on that in the first must be the same as the effect of the efflux in the second quadrant on that in the first— that is, the conductivity of the first semicircular disk into the upper semi-infinite atmosphere is diminished by the presence of the active half disk in the second quadrant to the same extent as the conductivity of the first semicircular disk into the right-hand semi-infinite atmosphere is diminished by the presence of the active half disk in the fourth quadrant.

The former has been shown to be $1/\sqrt{2}$ of its unimpeded value, hence with one face insulated the value of the conductivity of the half disk into the first and fourth

Fig. 5.
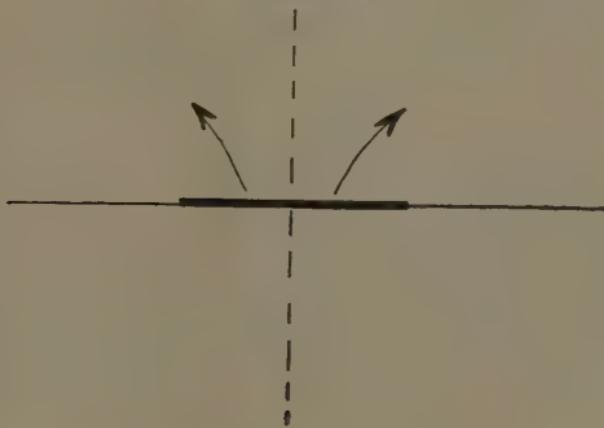
quadrants (fig. 3) is $2r\sqrt{2}$, and therefore for the complete disk with one face insulated in the complete infinite medium the conductivity is $4r\sqrt{2}$.

§ 6. Applying this conclusion to the case of the unflanged pipe, it is clear that the conductivity of the end of the tube is increased to $\sqrt{2}$ of its original value by the removal of the flange, and that the end-correction is $1/\sqrt{2}$ of that of the flanged tube. It follows that the correction for a flanged tube of radius r has the same correction as an unflanged tube of radius $r\sqrt{2}$, a conclusion which can be easily verified with tubes of $2\frac{1}{8}$ in. diameter (flanged) and 3 in. (unflanged) respectively.

$$[2\tfrac{1}{8}\sqrt{2}=3\cdot00.]$$

§ 7. The value of the correction for the flanged pipe calculated by Rayleigh from the expression

$$\text{conductivity} = \text{area/correction}$$

is $\pi r/4$, taking the conductivity to be $4r$. This is the lower limit. The conductivity calculated from the upper limit, $8r/3\pi$, is $3\pi^2 r/8$.

The geometric mean of these two values for the conductivity is $\pi r\sqrt{3/2}$, which gives the correction as $0.82r$, the actual value. Rayleigh explained the discrepancy in the former as due to the fact that the air vibration at the mouth of a pipe cannot be regarded as originating in a disk, and stated that the conductivity was actually larger. Taking the conductivity, then, for the flanged pipe as $\pi r\sqrt{3/2}$, removal of the flange modifies this to $(\pi r\sqrt{3/2})\sqrt{2}$, which is $\pi r\sqrt{3}$.

The end-correction calculated from this is $r/\sqrt{3} = 0.58r$.

References.

(1) Rayleigh, ' Sound,' ii. p. 172.
(2) King, Phil. Mag. xxi. p. 128 (1936).
(3) (a) Richardson, ' Sound,' 2nd ed., p. 164 ; (b) Wood, ' Sound,' p. 172.
(4) Rayleigh, *loc. cit.* pp. 180–182.
(5) Gray, Matthews, and MacRobert, ' Bessel Functions,' pp. 141–142.
(6) Rayleigh, *loc. cit.*
(7) Bate, Phil. Mag. xi. p. 629 (1930) ; Proc. Phys. Soc. xlviii. p. 100 (1936).

XLV. *On the Properties of certain Vibratory Doublets. By* E. TAYLOR JONES, *D.Sc., Professor of Natural Philosophy in the University of Glasgow* *.

A SUGGESTION recently made by the present writer † that electrostatic forces represent the variation of the mutual energy of electrons regarded as coupled oscillatory systems led to the result that the frequency of each of two electrons of opposite sign becomes zero when they approach to a certain critical distance

* Communicated by the Author.

† Phil. Mag. xxi. p. 337, and xxii. p. 921 (1936). The critical distance r_0 is equal to k/a, k being the coupling at unit distance, and a representing $4\pi^2 v_0^2$, where v_0 is the frequency of an electron at rest and far removed from other electrons.

r_0 ($=2 \cdot 8 \cdot 10^{-13}$ cm.) of each other, and remains zero at shorter distances. In one of the papers referred to * it was suggested that when two unlike electrons are within the distance r_0 of each other they disappear as such, but the possibility should be considered that they can continue to exist even though their frequency is zero. It is clear that in these circumstances the two electrons do not exert electrostatic attraction upon each other, and that if they are at rest and are " free," *i. e.* far removed from other electrons, their energy and mass will also be zero. Some other properties of these doublets, which have zero energy when free and at rest, will be considered in the present paper.

The Common Frequency of Two Stationary Electrons at various distances apart.

The expression for this frequency, in the case of two electrons of opposite sign at distance r † ($r > r_0$), is

$$\nu = \frac{1}{2\pi} \sqrt{a - \frac{k}{r}}, \quad \ldots \ldots (1)$$

and its value is shown by the lower curve of fig. 1 ‡, the abscissa being r/r_0, and the ordinate $2\pi\nu/\sqrt{a}$. It is seen that the frequency is zero over the range $r = 0$ to r_0, that at greater distances it increases with the distance, at first with infinite rapidity, finally reaching the " free " value $\sqrt{a}/2\pi$ when the two electrons are infinitely far apart.

The force of attraction between the electrons is indicated by the slope of the curve. It is zero over the range 0 to r_0, is infinite immediately beyond this range, and it diminishes gradually to zero as the distance further increases. The expression for the force, if $r > r_0$ (*l. c.* p. 341), is

$$F = \frac{hk}{2\pi r^2 \sqrt{a}} \cdot \frac{1}{\sqrt{1 - k/ar}}. \quad \ldots \ldots (2)$$

* *L. c.* p. 342.
† *L. c.* p. 341.
‡ The upper curve in fig. 1 shows the frequency of two electrons of the *same* kind at various distances apart, given by $2\pi\nu/\sqrt{a} = \sqrt{1 + k/ar}$. The frequency is constant and equal to $\sqrt{2a}/2\pi$ over the range $r = 0$ to r_0, diminishing to the " free " value $\sqrt{a}/2\pi$ at an infinite distance.

The two electrons forming a doublet at rest, therefore, while free from mutual attraction or repulsion, cannot be separated to a greater distance apart than r_0, since this would require an infinite force. Being inseparable in this sense they may be regarded as a single particle of zero mass and energy.

Fig. 1.
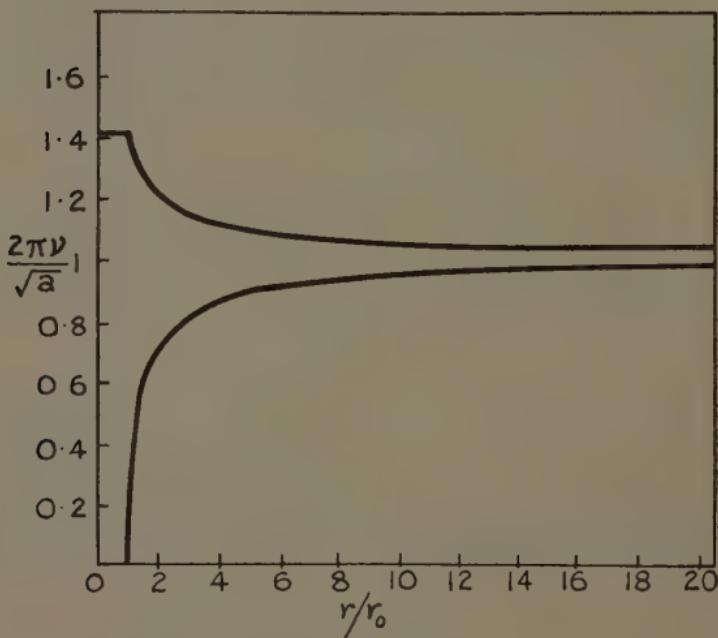
Frequency of two electrons of same sign (upper curve) and of opposite sign (lower curve) at various distances apart.

The Direct Approach of two Electrons of opposite sign.

The question arises whether a doublet can be formed by the direct approach of a positive and a negative electron. The expression k/r for the electrostatic coupling of two stationary electrons will also hold for two electrons moving with the same velocity in the same straight line, r being the distance between the electrons as it appears to an observer moving with them. To a stationary observer, however, the distance r will appear shortened, and the electrostatic coupling becomes $k\sqrt{1-v^2/c^2}/r$. There is no reason for supposing that the electrostatic coupling will be different if the velocity of one of the

electrons is reversed *, and there is no magnetic action since they are moving in the same straight line, so that the energy equation for two unlike electrons approaching each other directly with equal speeds v is

$$a - \frac{k\sqrt{1 - v^2/c^2}/r}{1 - v^2/c^2} = a - \frac{k}{r_1}, \quad . \quad . \quad . \quad (3)$$

r_1 being the distance apart at which the electrons start from rest. If r_1 is infinite or has any finite value greater than r_0 the right-hand side of (3) is finite, and if the electrostatic coupling becomes equal to a this can only happen if $v = c$, and therefore also $r = 0$. The motion of the electrons is accelerated until they meet, when their speed reaches the value c. The energy, however, is finite and constant throughout the motion, being equal to the initial energy, *i. e.*, $2h\nu_0$ if the approach is from an infinite distance, in which case this must also be the energy of the products of the collision †. The critical distance, within which the coupling is equal to a, shrinks to zero in the line of motion, and it is not probable that a doublet could be formed by the direct approach of the two electrons. It is clear that a doublet cannot move with the velocity c and with one of its component electrons ahead of the other, since in such circumstances the electrons would have zero coupling and infinite energy. If a doublet is moving with the velocity of light its two electrons must be moving at right angles to the line joining them.

The Energy of a Moving Doublet.

A doublet being treated as a single particle its energy, when it is moving transversely with velocity v, is

$$h\nu = \frac{h}{2\pi} \sqrt{\frac{a - b}{1 - v^2/c^2}}, \quad . \quad . \quad . \quad (4)$$

b being the electrostatic coupling. Since $b = a$ the frequency and the energy are zero unless $v = c$. The

* According to electromagnetic theory the field is the same behind and in front of a moving charged particle.

† What the result of the collision may be cannot be decided without further knowledge of the nature of an electron. It is conceivable that the two electrons interpenetrate each other, and afterwards separate as electrons, but it has also been suggested that two colliding electrons become changed into particles of other kinds.

frequency of a doublet moving with velocity c may have any value, being determined by the initial conditions and the manner in which the doublet is set into motion. If, for example, two similar doublets are set into motion by their mutual repulsion from rest in the relative positions shown in fig. 2, the frequency of each doublet will be constant during the motion, and can be calculated from the frequency of the system in its initial position. The distance between the two electrons of each doublet being taken to be just within the limiting value r_0,

Fig. 2.

$+ \bullet$ $\qquad\qquad\qquad\qquad$ $\bullet +$

$\longleftarrow\!\!\!\rule{2cm}{0.4pt} s_0 \rule{2cm}{0.4pt}\!\!\!\longrightarrow$

$- \bullet$ $\qquad\qquad\qquad\qquad$ $\bullet -$
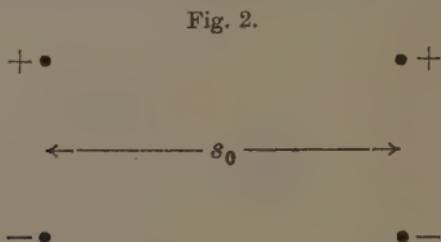
and the distance between the doublets being s_0, the frequency of the system is given, with $n = 2\pi\nu$, by

$$\frac{n^2}{a} = \frac{r_0}{s_0}\left(1 - \frac{1}{\sqrt{1 + r_0^2/s_0^2}}\right), \text{ if } s_0 > r_0, \quad . \quad . \quad (5)$$

$$\frac{n^2}{a} = 1 - \frac{1}{\sqrt{1 + s_0^2/r_0^2}}, \text{ if } s_0 < r_0. \quad . \quad . \quad . \quad (6)$$

Values of $n/\sqrt{a}$ calculated from (5) and (6) are shown plotted against the ratio s_0/r_0 in fig. 3.

It appears from the graph that when the two doublets are extremely close together ($s_0/r_0 = 0$) the frequency, and therefore also the energy, of the system is zero *, although the electrons of similar sign are in close contact with each other. As s_0 increases from zero the frequency and energy increase rapidly to a maximum at $s_0/r_0 = 1$, the value of $n/\sqrt{a}$ at this distance being 0·5413. Over the range $s_0 = 0$ to r_0, therefore, the two doublets attract each other, the force of attraction (indicated by the slope of the early part of the curve in fig. 3) falling off slightly as s_0/r_0 approaches unity. The physical reason

* It can be shown that if there are any number of electrons of one sign and at least one electron of the opposite sign within a sphere of diameter r_0, all being stationary and far removed from other electrons, the common frequency of the system is zero.

for this attraction is, of course, that within this range the particles of like sign are at less than the critical distance r_0 of each other, and therefore do not repel each other, while electrons of opposite sign are beyond the critical range and therefore attract each other. The value of the attraction at any distance s_0 less than r_0 is, by (6),

$$\mathrm{F} = 2h\,\frac{d\nu}{ds_0} = \frac{h\sqrt{a}\,.\,s_0}{4\pi r_0^2}\left(1+\frac{s_0^2}{r_0^2}\right)^{-\frac{3}{2}}\left(1-\frac{1}{\sqrt{1+s_0^2/r_0^2}}\right)^{-\frac{1}{2}},$$
. . . (7)

Fig. 3.
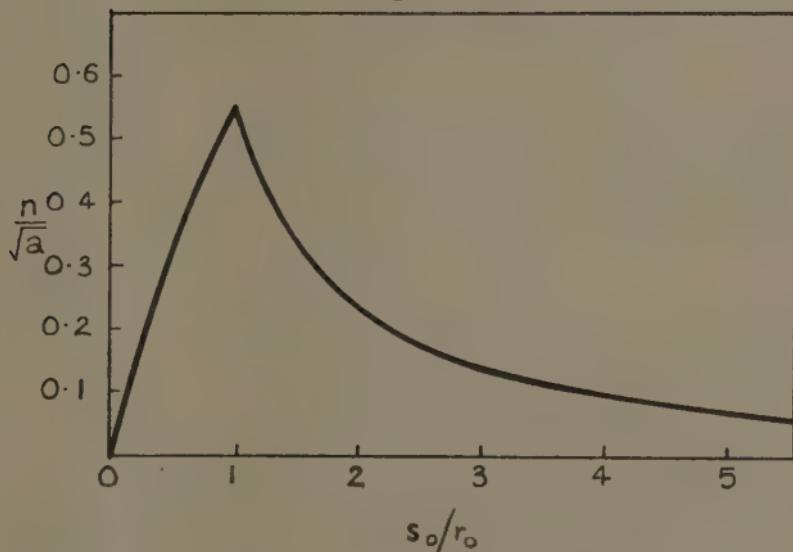
Frequency of two parallel doublets at various distances apart.

which at $s_0 = 0$ has the finite value

$$\mathrm{F}_0 = \frac{hk\sqrt{2}}{4\pi r_0^2\sqrt{a}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Inserting the numerical values $h = 6.55 \cdot 10^{-27}$, $k = 1.7 \cdot 10^{29}$, $r_0 = 2.8 \cdot 10^{-13}$, $a = 6.07 \cdot 10^{41}$, all in c.g.s. units (*l. c.* p. 340), we find for the force of attraction between the two doublets, when stationary and extremely close together, the two electrons of each doublet being just within the distance r_0 of each other, the value

$$\mathrm{F}_0 = 2.05 \cdot 10^6 \text{ dynes.} \quad . \quad . \quad . \quad . \quad (9)$$

As the distance between the doublets increases beyond r_0 the frequency falls off from the maximum, rapidly at first, more slowly at greater distances, ultimately

becoming zero at very great distances, at which the doublets are free from each other's influence. The mutual repulsion of the doublets for any value of s_0 greater than r_0 is found from (5), its greatest value, at $s_0 = r_0$, being

$$F_m = \frac{h\sqrt{a}}{2\pi r_0}\left(1 - \frac{1}{\sqrt{2}}\right)^{-\frac{1}{2}}\left(1 - \frac{1}{2\sqrt{2}}\right)$$

$$= 2{\cdot}23 \, . \, 10^6 \text{ dynes.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The maximum repulsion between two doublets having their axes at right angles to the line joining them is therefore rather greater than the maximum attraction *.

If the two doublets are at rest at any distance s_0 apart $(s_0 > r_0)$ and are free to move they will separate with equal velocities and accelerations, and soon move freely with velocity c and with frequency equal to the initial value at s_0 as shown by the curve of fig. 3. The energy equation for this motion reduces to

$$\frac{f\left(\dfrac{s}{\sqrt{1 - v^2/c^2}}\right)}{1 - v^2/c^2} = f(s_0), \quad . \quad . \quad . \quad (11)$$

where

$$f(s) = \frac{1}{s} - \frac{1}{\sqrt{s^2 + r_0^2}}.$$

The greatest frequency which free doublets can have if set into motion in this way is that corresponding to the position $(s_0 = r_0)$ of maximum energy in fig. 3, *i. e.*, $0{\cdot}5413 \, \nu_0 = 6{\cdot}71 \, . \, 10^{19}$. If the initial distance s_0 is $10 \, r_0$ the frequency is $2{\cdot}63 \, . \, 10^{18}$.

The motion produced in this way is, however, not reversible. If the two freely moving doublets are reflected so as to approach each other directly with velocity c they cannot exert any retarding force upon each other, since the coupling is reduced to zero by the motion, and they cannot be brought to rest in their initial positions. A doublet moving with the velocity of light can, however,

* The corresponding expressions when the two electrons of each doublet are at any distance d apart $(d < r_0)$ show that the frequency is again a maximum at $s_0 = r_0$, and is zero over the range $s_0 = 0$ to $\sqrt{r_0^2 - d^2}$. The frequency and the mutual force are, of course, zero at all values of s_0 if $d = 0$. If one of the doublets in fig. 2 is replaced by a pair at a considerably greater distance d' apart $(d' > r_0)$, the frequency of the system approximates to the free frequency of this pair, *i. e.*, $\sqrt{a - d'}/2\pi$.

be slowed down by passing it between systems of fixed charges so disposed that the electrostatic energy of the doublet and the charges is increased at the arrival of the doublet. In such circumstances the total energy of the latter must be diminished, since some of it has been imparted to the fixed charges, and the velocity of the doublet must therefore also be diminished. It is not difficult to conceive of arrangements of fixed charges by which a doublet may be brought to rest or reflected in this way, and, in fact, the setting of a doublet into motion with the velocity of light, and the stopping or reversing of this motion, appear to be processes which might frequently occur in nature.

The Doublet and the Photon.

From the above examples and considerations it is obvious that several of the properties of a doublet moving with velocity c are similar to those generally ascribed to the photon. The fact that a free doublet has no energy or momentum unless it is moving with the velocity of light, the transverse position of the electrical axis of a moving doublet, and the dependence of the frequency of a doublet upon that of the source by which it is set into motion all have corresponding features in any theory of the photon. The property of a moving electron of being scattered by atoms of matter, and of giving rise under suitable circumstances to observable diffraction phenomena, can reasonably be assumed to be possessed also by a doublet. It can be shown that the wave-length $\lambda(=h/mv)$ of the de Broglie waves associated with a moving doublet is given by the equation

$$\frac{1}{\lambda^2} = \frac{\nu^2 - \nu_1{}^2}{c^2}, \quad \ldots \ldots \quad (12)$$

ν_1 being the frequency of the doublet when at rest. In the case of a free doublet $\nu_1 = 0$, and the relation between wave-length and frequency is $\lambda = c/\nu$, as it is for a photon.

It may be supposed that in an enclosure there are doublets which are at rest and have no energy, others which are at rest or moving with velocity less than c and have electrostatic energy owing to their presence near other charges (*e. g.*, in the walls of the enclosure), and a third class which are moving freely with velocity c and have energy depending upon their frequency

The intermediate stage may, however, be regarded as transitory, merely marking the beginning or end of free movement, and not occupying an appreciable fraction of the time. The energy, therefore, is confined to those doublets which are moving freely with the velocity of light, and since free interchange can take place between these and the doublets of zero energy they cannot be regarded as fixed in either identity or number. The total energy of the doublets in a given volume at a given temperature may be regarded as constant, and in these circumstances the distribution of energy among them when the system is in equilibrium must be that given by the statistical method of Bose and represented by Planck's radiation formula.

On the other hand, there are certain obvious differences between the properties of a vibratory doublet in motion and those known to be possessed by a photon, such as that of being deflected in a suitably applied non-uniform electric field. This effect would be small in experimental conditions, and so far as the writer is aware there are no known facts which are definitely contrary to the view here suggested that a photon is a vibratory doublet in translational motion. This theory is based upon the presumed fact that an electron has the characteristics of an oscillator, and as to this there is strong evidence derived from experiment.

Natural Philosophy Department,
The University, Glasgow.
July 1937.

XLVI. *On the Atomic Forces of Solid State.* *By* Weng Wen-Po *.

Part VI.

(18) NON $METALS$.—The atomic force in a solid body was represented by us, using two functions, $f(r)$ and $F(x)$ †. $f(r_0+x)$ is the force acting on an atom due to all the neighbouring atoms to one side of the yz-plane when the atom is displaced along the x-axis by an amount x from its equilibrium position. r_0 is the average distance

between two neighbouring atoms, each at its respective equilibrium position. $F(x)$ was defined as the sum of $f(r_0-x)$ and $f(r_0+x)$ which represents the restoring force of the atom. The function $f(r_0+x)$ or $f(r)$ was represented by converging series as $f(r_0+x) = -ax + bx^2 - cx^3$ when x is small compared with r_0.

By assuming $\dfrac{br_0}{a}$ to be approximately a constant, we were able to explain Lindemann's and Grüneisen's formulæ for atomic vibrational frequency. This was done by assuming that at the melting point the displacement is given approximately by the equation $\dfrac{df(r)}{dr} = 0$, and that the thermal expansion between two neighbouring atoms is $\varDelta x = \dfrac{1}{2}\dfrac{b}{a}X_1^2$, when the atom is oscillating with a maximum amplitude X_1.

By certain considerations which we will discuss in the next section, $\dfrac{br_0}{a}$ is expected to have the approximately same numerical value for non-metals as for metals and their compounds. So both Lindemann's and Grüneisen's formulæ will also hold good for them. To compare the value of frequency so obtained, we have available Debye's third-power law of specific heat and Madelung-Sutherland-Einstein formulæ of compressibility which was roughly proved in the previous works to be independent of the nature of substance. In Table XVII., the values of ν, the characteristic atomic vibrational frequency of non-metallic elementary solids, calculated from various physical data, are tabulated. In the second column, the values of ν are those which have been calculated from the specific heat at low temperatures by other authors. The values in the third column are estimated from the compressibility data by the equation $\nu = 2 \cdot 9 . 10^7 . \rho^{-\frac{1}{6}} W^{-\frac{1}{3}} \beta_0^{-\frac{1}{2}}$, where ρ is the density, W the atomic weight, and β_0 the compressibility. Lindemann's formula in the form $\nu = 2 \cdot 8 . 10^{12} . \rho^{\frac{1}{2}} T_m^{\frac{1}{2}} W^{-\frac{5}{6}}$, where T_m is the melting point on the absolute scale, gives the values in the fourth column. If the quantization of energy is taken into consideration after Debye's consideration, then

$$\nu = 3 \cdot 0 \times 10^{12} . \sqrt{D\left(\frac{\Theta}{T_m}\right)} . T_m^{\frac{1}{2}} \rho^{\frac{1}{2}} W^{-\frac{5}{6}},$$

where $k_0\Theta = h\nu$ and $D\left(\dfrac{\Theta}{T_m}\right)$ is Debye's energy function

$$D\left(\frac{\Theta}{T}\right) = 3\left(\frac{T}{\Theta}\right)^3 \times \int_0^{\frac{\Theta}{T}} \frac{y^3 dy}{e^y - 1}.$$

The calculated values are tabulated in the 7th column. Grüneisen's formula in the form of

$$\nu = 1\cdot6.\,10^{11}.\,W^{-\frac{1}{3}}\rho^{\frac{1}{6}}S_h(T)^{\frac{1}{2}}\alpha_l(T)^{-\frac{1}{2}},$$

where $S_h(T)$ is the specific heat in calories per degree per gram at a fairly low temperature T and $\alpha_l(T)$, the coefficient of linear thermal expansion at the same tempera-

TABLE XVII.

	$\nu \times 10^{-12}.$ Sp. ht.	$\nu \times 10^{-12}.$ Compr.	$\nu \times 10^{-12}.$ Melting point.	$\nu \times 10^{-12}.$ Thermal expansion.	$\nu \times 10^{-12}.$ Surface tension.	$\nu \times 10^{-12}.$ Melting point quantized.
A	1·8*	....	1·40	....	1·45	1·2
Br	1·8†	....	1·78	....	1·75	1·8
Cl	2·3†	....	2·32	....	2·43	2·2
I	1·6†	1·2	1·65	1·3	....	1·7
N	2·0*	....	2·5	....	2·25	1·8
O	2·3*	....	2·33	....	2·8	2·3
Ne	....	....	1·0	....	1·2	0·7
He	·5–·8‡	·4	....	....	·8	
Si	10·0§	14	9·56			
F	....	....	2·39	....	....	1·6
C$_{dia.}$......	39	3 × 10	....	4 × 10		
B	....	21	26			
H	2·0‖	....	4·5	....	4·4	1·2
CO	2·5*	....	....	....	2·4	

ture is used for the computation of ν listed in the fifth column.

A semi-empirical formula $\nu = 2\cdot6.\,10^{12}.\,S_m^{\frac{1}{3}}W^-$, where S_m is the surface tension appropriate to the melting point, also yields satisfactory results as listed in the sixth column. The fact that the constant $2\cdot6$ is larger than that of the corresponding formula for metals indicates the difference in the structure. It was found that the equation $\nu = 2\cdot6.\,10^{12}.\,S^{\frac{1}{2}}W^{-\frac{1}{4}}$ holds good also for many binary compounds such as CO, etc.

* *Verh. d. Phys. Ges.* xviii. p. 4 (1916).
† *Zeits. f. Phys. Chem.* cxii. p. 467 (1924).
‡ *Rev. Mod. Phys.* viii. p. 352 (1936).
§ *Sitz Ber. Akad. Wiss.* mcmxiv. p. 355 (1914).
‖ *Zeit. f. Phys. Chem.* xv. Heft 2/3, p. 121 (1931).

In Table XVIII. are listed the constants used in computing Table XVII.

The data from the Raman effect and Infra-red spectra are still inadequate to give any definite conclusion, though they would be very helpful. Sulphur has no single dominant frequency. Its specific heat curve seems too flat for a single frequency *, so it is left out of the table.

The facts here presented should be useful for the estimation of physical constants which are difficult to measure, besides their indication of the similarity existing in the structure of solids.

TABLE XVIII.

	$T_m°A$.	ρ.	W.	$\beta_0 \times 10^{12}$.	S_m.	$\alpha_l \times 10^6$.
A	84·1	1·65	39·91	....	12·5	
Br	266·0	3·4	79·92	....	36	
Cl	171·6	1·9	35·46	....	31·2	
I	386·7	4·93	126·93	13·2	....	93
N	63·3	1·026	14·01	....	10·53	
O	54·8	1·426	16·00	....	18·3	
Ne	24·6	1·0	20·2	....	5·90	
He	....	....	4·0	1500	0·353	
Si	1693	2·4	28·06	0·325		
F	50·2	1·3	19·00			
C_{dia}........	....	3·51	12·00	0·16	....	0·9
B	2570	2·3	10·83	0·3		
H	14·1	0·081	1·01	....	2·88	
CO........	....	....	....	....	12·11	

The introduction of quantization of energy in the equation,

$$\nu = 3·0 \times 10^{12}. \sqrt{D\left(\frac{\Theta}{T_m}\right)}. T_m^{\frac{1}{2}}\rho^{\frac{1}{3}}W^{-\frac{5}{6}}, \quad . \quad . \quad (56)$$

makes the calculated value of ν for hydrogen somewhat nearer to that calculated from specific heat. By equation (56) we are able to estimate the melting point of solid helium under no external pressure. Transposing equation (56), we obtain

$$T_m D\left(\frac{\Theta}{T_m}\right) = \left(\frac{\nu . 10^{-12}}{3·0}\right)^2 \frac{W^{\frac{5}{3}}}{\rho^{\frac{2}{3}}},$$

which is equal to 0·8 if $\rho = 0·2$ and $\nu = 0·5 . 10^{12}$. At low

* *Ann. d. phys.* xxxvi. p. 395 (1911).

temperature compared with Θ, $D\left(\dfrac{\Theta}{T_m}\right) = 19.\left(\dfrac{T_m}{\Theta}\right)^3$. Thus

the melting point of helium at zero external pressure becomes 5° absolute. It is probably the zero point energy which prevents freezing.

(21) *General Theory.*—The fact that $\dfrac{br_0}{a}$ is nearly a

constant for all solids can be easily explained in many ways. But we would like to investigate the matter in the reverse order, *i. e.*, to find what explanations are possible. This can be done by expanding the function $f(r)$ into a Taylor's series as follows :

$$f(r) = f(r_0 + x) = f(r_0) + \left[\frac{df(r)}{dr}\right]_{r=r_0} x + \frac{1}{2}\left[\frac{df(r)}{dr}\right]_{r=r_0} x^2 + \ldots,$$

where $r = (r_0 + x)$. According to our assumption, also

$$f(r) = f(r_0 + x) = -ax + bx^2 - cx^3.$$

By comparing the coefficients of x, x^2, etc., we obtain

$$\left.\begin{aligned}
-a &= \left[\frac{df(r)}{dr}\right]_{r=r_0}, \\
b &= \frac{1}{2}\left[\frac{d^2f(r)}{dr^2}\right]_{r=r_0}, \\
-c &= \frac{1}{6}\left[\frac{d^3f(r)}{dr^3}\right]_{r=r_0},
\end{aligned}\right\} \quad \ldots \quad (57)$$

and

$$b' = \frac{br_0}{a} = -\frac{r_0}{2}\frac{\left[\frac{d^2f(r)}{dr}\right]_{r=r_0}}{\left[\frac{df(r)}{dr}\right]_{r=r_0}}, \quad \ldots \quad (58)$$

$$c' = \frac{br_0^2}{a} = \frac{r_0^2}{6}\frac{\left[\frac{d^3f(r)}{dr^3}\right]_{r=r_0}}{\left[\frac{df(r)}{dr}\right]_{r=r_0}}. \quad \ldots \quad (59)$$

Now, for all elementary solids, b' is nearly a constant. So an explanation can be found by solving equation (58), with b' considered constant. The solution is

$$f(r) = A_1 + \frac{B}{r^q}.$$

Considering $f(r_0) = 0$, the solution can be generalized as

$$f(r) = \frac{A}{r^p} + \frac{B}{r^q}, \quad \ldots \ldots \quad (60)$$

where A_1, A, B, p, and q are constants, including zero and negative values. This means that the force acting between atoms of any kind consists of two parts, each of which is proportional to a certain power of the atomic distance so long as the displacement is not large.

It is most interesting that equation (60) is just the form commonly accepted, which represents a repulsion and an attraction term.

By definition, $f(r_0) = 0$ and $\dfrac{A}{r_0^p} = \dfrac{B}{r_0^q}$, solving equations (58), (59), and (60), we can express $\dfrac{br_0}{a}$ and $\dfrac{cr_0^2}{a}$ in terms of the constants p and q:

$$b' = \frac{br_0}{a} = \frac{p+q+1}{2}, \quad \ldots \ldots \ldots \quad (61)$$

$$c' = \frac{br_0^2}{a} = \tfrac{1}{6}[q^2 + (p+3)q + (p+2)(p+1)]. \quad (62)$$

If the index of the repulsion term, say q, is much higher than that of the attraction term p, $\dfrac{br_0}{a}$ depends mainly on the value of q. That the inverse repulsive indices of various atoms are high and about the same, is enough to explain some of the relations between physical constants.

From the equations $f(r) = \dfrac{A}{r^p} + \dfrac{B}{r^q}$ and $f(r_0) = 0$ we are able to eliminate either A or B. If we eliminate B and expand $f(r)$ in a Taylor's series,

$$f(r) = \frac{A(p-q)}{r_0^p}\left[-\left(\frac{x}{r_0}\right) + \frac{p+q+1}{2}\left(\frac{x}{r_0}\right)^2 - \ldots \right].$$

Comparing with

$$f(r) = ar_0\left[-\left(\frac{x}{r_0}\right) + b'\left(\frac{x}{r_0}\right)^2 - \ldots \right],$$

we get

$$A(p-q)\tfrac{1}{4}ar_0{}^{p+1}. \quad . \quad . \quad . \quad . \quad . \quad (63)$$

In section (10) of the last paper, we discussed the quantity a' which is proportional to $ar_0{}^2$, this would be proportional to $A(p-q)$ if $p=2$.

So far we have dealt with the similarity of the atomic forces of various sorts of atoms. It does not necessarily follow that the expression $f(r) = \dfrac{A}{r^p} + \dfrac{B}{r^q}$ is exact, even within a limited range of atomic motion. On the contrary, the former expression $f(r) = -ax + bx^2 - cx^3 + \ldots$ will represent the facts much better if sufficient terms are taken.

If we take $\dfrac{df(r)}{dr} = 0$ at melting point, the displacement of the atom X, at the melting point greatly depends on the form of $f(r)$ assumed. If $f(r) = -ax + bx^2$ is taken, as in section (11)*,

$$\left(\frac{X_1}{r_0}\right) = \frac{1}{2}\left(\frac{a}{br_0}\right) = \frac{1}{2b'}.$$

When $f(r) = \dfrac{A}{r^p} + \dfrac{B}{r^q}$ is assumed,

$$\left(\frac{X_1}{r_0}\right) = \left(\frac{q}{p}\right)^{\frac{1}{q-p}} - 1.$$

So the values of X_1 can be widely different through the same values of p and q are taken. But nevertheless both values are approximately constant if q is. This gives rise to no difficulty in the explanation of Lindemann's equation.

As a matter of fact, the values of q, and therefore $\dfrac{br_0}{a}$, are not quite constant. This explains why both Lindemann's and Grüneisen's formulæ are not accurate.

XLVII. *The Effect of Discontinuities of the Background on the Evaluation of the Intensities of X-ray Reflexions from Crystalline Powders. By* A. BAXTER, *Ph.D., and* J. C. M. BRENTANO, *D.Sc.**

Summary.

THE paper relates to the measurement of X-ray intensities ; it discusses in particular the discontinuous character of the background of white radiation and its effect on the evaluation of the intensities of X-ray reflexions. Discontinuities of two types are considered arising from absorption steps and from sudden sensitivity changes of the photographic emulsion for certain wavelengths. Means of avoiding errors arising from background steps situated close to reflexion lines are discussed. It is shown that the height of the background steps in relation to the height of the lines decreases with improved definition, and that X-ray reflexions should be recorded under conditions of good focusing. With regard to powder photographs, reference is made to a substitution method of a type developed by one of the writers with which the reflexions from different substances are recorded separately and the crowding of lines and of background steps is reduced as compared with the method of mixed powders. It is shown that the former method is useful for qualitative comparisons and for evaluating intensities of reflexions from different preparations of the same substance, but that its application to intensity measurements is limited by the fact that the values of the absorption coefficients should be known to a high degree of accuracy, so that it cannot replace the latter in a general way. Other means of eliminating some of the background steps by compensating filters are discussed. A test making use of two wave-lengths for ascertaining the state of a powder mixture is indicated. A technique is advocated of deriving intensity data not from averaging measurements of a great number of lines, but from measurements of selected lines found free from the interference of background steps.

* Communicated by Dr. J. C. M. Brentano.

QUANTITATIVE measurements of X-ray intensities have in recent times assumed increased interest, since a better knowledge of the atomic scattering factors and of their changes with wave-length has made it possible to use quantitative data in deciding finer points of crystal-structure analysis. Such measurements are conveniently made with powdered crystals, for if the particles are sufficiently small, the complications of extinction effects can be avoided. Powders offer the further advantage that measurements from a large number of spacings can readily be obtained and that measurements from various crystals can be placed on a common scale by admixture of a comparison substance.

In the present paper we wish to draw attention to the effect upon such intensity measurements of certain discontinuities in the intensity distribution of the white radiation which generally accompanies the monochromatic radiations used, and which appears as a " background " to the reflexions of monochromatic emission lines. The problem arose as a necessary preliminary to measurements of the change of atomic scattering factors of heavy atoms with wave-length for radiations of frequencies close to their L absorption frequencies *, and no consistent results could be obtained until it was solved. The considerations, however, apply equally well to all intensity measurements in which great accuracy of the individual determination is required. In our work we used photographic recording; similar conditions are encountered with ionization-chamber methods, to which our results apply with only slight modifications.

1. The typical conditions are exemplified by fig. 1, in which the photographic densities or ionization goniometer readings translated into X-ray intensities are plotted as ordinates against an angle or equivalent coordinate of position as abscissa. Each reflexion of a monochromatic " line " is represented as an elevation; its total or " integrated " intensity is determined by extrapolating the background from each side to form a base-line and measuring the area contained between this base-line and the contour of the elevation. Irregularities of the background are usually treated as accidental, and a smooth curve through them is taken as base-line. The actual

* J. Brentano and A. Baxter, *Zeits. f. Phys.* lxxxix. p. 720 (1934).

contour of the elevation representing a line depends on a number of factors such as the limited definition of the goniometer, the size of the scattering crystallites etc In so far as these factors do not comply with the conditions of perfect definition, some more or less extended broadening of the curve towards the base always occurs. Because of this broadening any error in the correct location of the base-line produces a much greater change in the area enclosed than would be produced by the same error if it occurred in locating the curve nearer to the peak, and so the conditions determining the correct

Fig. 1.
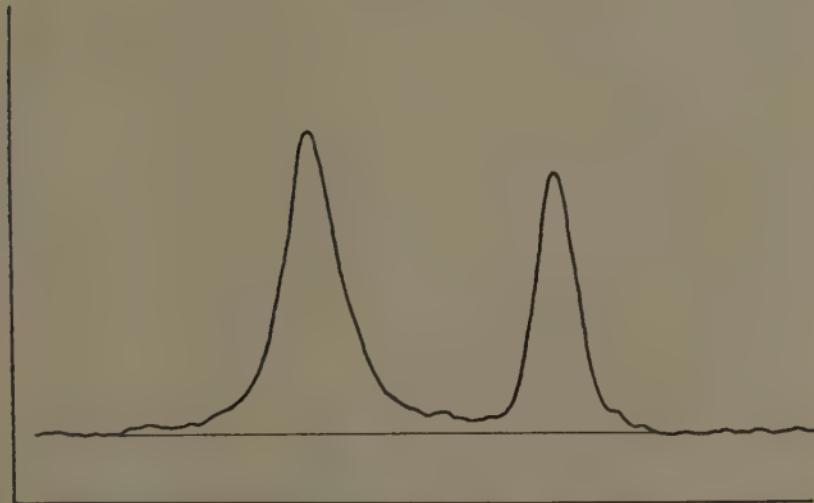
Example of microphotometry record ; the " base-line " is drawn.

location of the base line are of considerable importance in quantitative intensity measurements.

The reflexions from one set of lattice planes of spacing d of a single crystal occur at glancing angles θ given by the equation $2d \sin \theta = n\lambda$, where λ is the wave-length of the radiation and n an integer between *one* and a greatest value n' depending on the wave-length and determined by the condition that $\sin \theta$ must be less than unity. Any particular wave-length is therefore reflected to n' angles, where n' depends on that wave-length. For a continuous range of wave-lengths represented by white radiation we obtain in this way so many angular ranges corresponding to so many orders, possibly overlapping,

but such that each angle is associated with definite wave-lengths, and each wave-length with definite angles. Any singularity in the white radiation, *e. g.*, a sudden change in intensity at a certain wave-length, will thus be associated with definite angles in the same way as is an emission line.

While the distribution of white radiation with wave-length is initially a smooth and continuous function extending to a shortest limit determined by the potential difference applied to the tube, absorption of the rays in matter produces discontinuities at the particular wave-lengths which correspond to the absorption edges characteristic of the atoms of which the absorbing materials are constituted. We have thus an abrupt change of intensity for the wave-length corresponding to a K absorption edge where the absorption coefficient changes in the ratio of approximately 8, and lesser, but still considerable changes for the L and higher edges. For each such discontinuity n' steps in the background will appear at definite angles.

The number of such steps is determined by those absorption edges belonging to the constituents of the materials traversed by the rays which fall within the range of wave-lengths of the white radiation.

In general we have one or several absorption edges associated with the material of the anticathode itself ; for instance, in the case of a copper or iron anticathode an absorption edge situated somewhat to the short-wave side of the CuK_α or FeK_α radiations used. We have additional absorption steps due to any material deposited on the anticathode—for instance, sputtered tungsten in the case of a tungsten filament tube.

Further, we have the absorption edge of the filter generally used to suppress the unwanted components of the group of monochromatic radiations emitted by the anticathode—for instance, the Ni filter used with a Cu anticathode—which is of such a thickness that it reduces the short-wave K_β and K_γ components to a negligible amount, while leaving the K_α components sufficiently strong. This filter is deliberately chosen so as to have a very marked selective effect, and it produces a very pronounced step in the intensity of the white radiation. Similar steps are added by the various constituents of the reflecting crystal.

In our dispersion measurements a further filter had to be placed between the specimen and the photographic film in those cases where the monochromatic radiation used was of so short a wave-length as to excite the fluorescent radiation of the atoms constituting the crystals *.

2. In addition to the discontinuities discussed above, there are others caused by the different sensitivity of the photographic emulsion to X-rays of different wave-lengths.

The effect consists in a sudden change of sensitivity of the photographic emulsion for the wave-lengths corresponding to the absorption edges of silver and bromine, with an increase of sensitivity of the order of one to four on the short-wave side of the absorption edges. If these absorption edges fall within the range of wave-lengths of the white radiation emitted by the tube, discontinuities in the background arise at those angles to which the wave-lengths corresponding to the absorption edges are reflected. It should be noted that while the absorption steps previously discussed constitute a transition from smaller to higher densities in passing from shorter to longer wave-lengths, the sensitivity steps here discussed are reversed, having the denser side of the step on the side of short wave-lengths. Similar steps occur with an ionization chamber at the wave-lengths corresponding to the absorption edges of the atoms which provide the primary photoelectrons.

It will be seen that the total number of background steps produced by these various discontinuities is given by $\Sigma n'$, where the summation is taken over the particular values of n' associated with the various wave-lengths of the

* Where the wave-length of the monochromatic radiation used is shorter than that of an absorption edge of one of the constituents of the crystal, fluorescent radiation is excited which produces a dense background and obliterates the reflexion lines. This fluorescent radiation has then a longer wave-length than the incident monochromatic radiation ; it can thus not be suppressed by selective filtering, but only by a filter depending on the increase of absorption with wave-length. The absorption coefficient increasing only relatively slowly with wave-length, the filters entail a considerable absorption of the monochromatic radiation used, with a corresponding increase of exposure time, particularly so when the difference in wave-length between the incident characteristic radiation and the characteristic fluorescent radiation excited in the specimen is small. The procedure adopted was, however, the only way in which adequate records could be obtained.

edges which affect the intensity of the white radiation
or the sensitivity of the photographic emulsion.

3. If we replace the single crystal of spacing d, which
has been considered so far, by a crystal powder, each
of the many spacings associated with the crystal structure
gives a set of reflexions and of background steps.   The
total number of steps is then obtained by adding the
steps related to the various reflecting lattice spacings.
On the one hand the discontinuities are then less marked,
because each lattice spacing is responsible for only one
part of the background, so that each discontinuity is
rendered less prominent.   On the other hand, the in-
tensities of the different sets of reflexions and steps acquire
different relative values according to the multiplicity
factors p, which represent the number of different ways
in which the planes with a particular spacing occur in
the structure.

4. It should be noted that for reflexions from single
crystals, as well as from crystal powders, the discon-
tinuities of absorption and sensitivity introduce steps
in the background only in so far as it is due to white
radiation which has been subjected to a process of spectro-
graphic separation, and is thus reflected to definite angles
according to its wave-length.   To this, however, is added
radiation scattered incoherently (Compton scattering)
and any other radiation taking its origin from the reflecting
crystal or powder which is not spectrographically sepa-
rated.   This latter will comprise in particular any
fluorescent radiation excited in constituents of the
crystal or in the powder, and also radiation taking its
origin from other parts of the goniometer for which it
does not act as a spectrograph.   These additional contri-
butions to the background with a smooth intensity
distribution reduce the prominence of the steps ; they
do not, however, remove the disturbing effect which the
discontinuities of the background produce on the measure-
ment of the lines.   In fact the additional smoothly
distributed part of the background is superposed on the
steps and on the lines as well, and merely adds to the
general density, while the lines like the discontinuous
part of the background are only due to regularly reflected
radiation.   In the particular case of the crystal powder,
the intensities of the lines and of the steps depend on the

multiplicity factors p of the lattice spacings with which they are associated. Those absorption steps will particularly affect the measurements which are due to reflexions with large p values and which are so situated as to fall near to weak lines for which p is relatively small.

The sudden change of absorption with wave-length, which is of a similar sharpness to the spectroscopic sharpness of the emission lines, leads to background steps which are broadened mainly owing to the limited resolving powder of the X-ray goniometer and of the scattering crystals, and the same factors which determine the width of the lines determine also the width of the steps, so that both are of the same order. Reduced resolution will broaden the steps and make them less conspicuous ; it will at the same time broaden the lines, but while the final height of the background steps remains unaltered, it reduces the relative height of the lines, so that the discontinuities of the background become actually more important in relation to the lines when focusing and resolution are defective. This is so because the actual spectroscopic width of the lines is small compared with the width on the film which results from the limited definition of the goniomete · and of the reflecting crystals, while for the background the regions to the two sides of the step are large compared with the width of the transition band.

Fig. 2 illustrates some typical cases of how errors arise from the superposition of the lines and background steps, and shows the greater difficulty of obtaining correct evaluations of intensities with broadened reflexions. The fully drawn lines on the diagrams indicate the intensity distribution as observed, the dotted lines the intensity of the background as derived from known steps in absorption or in the sensitivity of the photographic plate. It will be seen that broad lines are more exposed to interference, owing to the greater chance of superpositions, and that any resulting error is increased because the height of the lines is reduced in comparison with that of the steps. In addition, where the broadening is particularly extended towards the base, a given displacement of the base-line causes a much greater change in the area enclosed than would the same displacement occurring in relation to a sharp reflexion line. To estimate the error which

may be incurred when a line and a background step are
not resolved, we must compare not only the relative
heights, but also the relative widths, or, more definitely,
the area representing the reflexion line as exemplified by
fig. 1 with the area representing the step falling in the
angular range covered by the line.   Primarily the intensity
of the line is determined by the intensity of the particular
characteristic radiation, and then by the scattering
factor F, by the multiplicity factor p for the particular
reflexion, by certain angular factors, and by factors

Fig. 2.
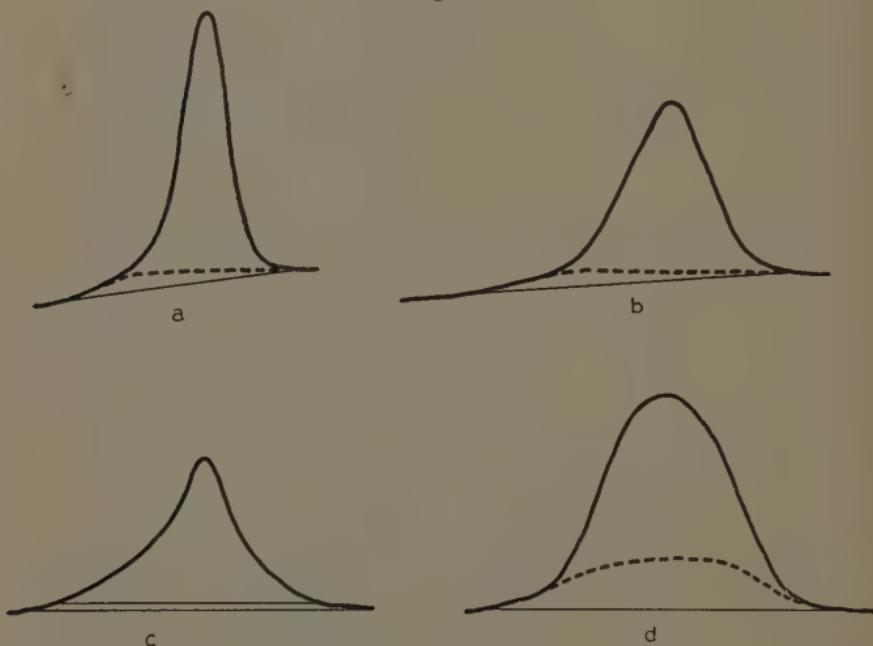
Types of errors in the location of the base-line.

(*a*) Background step just separated from reflexion line.
(*b*) Background step and reflexion line at the same angular distance
     as in (*a*); the definition is, however, reduced, so that, unless
     the step is known, its presence would certainly cause an error.
(*c*) Illustration of " vanishing-size distribution," one fraction of
     the particles being very small or deformed; these give a very
     broadened contribution to the reflexion.   The correct location
     of the base-line and evaluation of the area is then particularly
     difficult.
(*d*) Two steps of different type (absorption and sensitivity step) produce
     an inflexion of the base-line.   The actual area of the reflexion
     line is thus considerably smaller than would appear without
     considering this effect.   For examples of such steps see fig. 5.

depending on the setting of the goniometer and on the absorption coefficients. The magnitude of the step is determined by the difference in the specific intensities of the white radiation on the two sides of its edge, and further, by the F and p factors belonging to the reflexion producing it, and again by angular, instrumental, and absorption terms. For the case of overlapping these latter terms are the same for the line and the step, so that the essential data are known or can easily be determined. It will be noted that while the area representing the line is constant, the part representing the possible error introduced by the step is proportional to a term $\Delta i_\lambda \Delta\omega$, where Δi_λ is the difference in the specific intensities or film sensitivities for the two sides of the edge,

Fig. 3.

W
200

al
220

Ni abs step
from 220al

Ni abs.step
left from 211W.

W
211
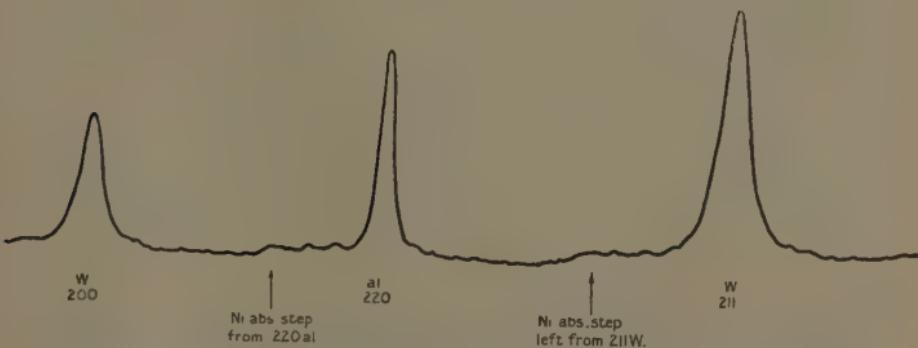
Record by scatter microphotometry of the reflexions of copper K radiation from a mixed powder of tungsten and aluminium. A nickel filter introduces steps in the background.

while $\Delta\omega$ stands for the width of the band into which the step is drawn or for the width of the line in case the latter determines the uncertainty in locating it. It will be seen that the possible error increases with the diffusion $\Delta\omega$.

5. In order to measure the magnitude of these effects experimentally, instances have been chosen where the line and step do not actually overlap, but are just separated enough for distinct measurement. Fig. 3 is a copy of a record obtained by scatter photometry from photographs of the reflexions of copper K_α radiation from a mixture of tungsten and aluminium powders. The nickel filter used to remove the unwanted K_β and K_γ components

of the characteristic radiation was so thick that no
measurable part of these components was transmitted,
and this filter introduced a single discontinuity into the
white radiation spectrum at its K absorption edge.   The
reflexions of this discontinuity appear as steps in the
background of the photometer records at the left of the
corresponding lines.   It can be seen that the step and the
line caused by reflexion from tungsten are both broadened
as compared with the step and line resulting from
reflexion at the aluminium lattice planes.   We should
point out that when scattering photometry is used the
correct position of the base-line is through the minima
of the irregularities in the photometer record, and not
through an average position.   That the steps are genuine
has been checked by the fact that they appear in the
same place on different photometer records and on
different photographs, and also by the fact that they
appear at the correct angles to which the wave-lengths
of the nickel absorption edge should be reflected.

In estimating the error which may arise from such a
step we have measured the area beneath the curve repre-
senting the (211) reflexion from tungsten, and also the
change which would be made in this area if the base-line
were displaced at one side of the curve by an amount
equal to the height of the step which results from the
reflexion of the nickel absorption edge from the same
(211) lattice planes of tungsten.   This change amounted
to 7 per cent. of the total area, and represents the error
which would be incurred if the step and line were just not
resolved and the existence of the step were ignored.
The same calculation for the step and line arising from
reflexion at the aluminium (220) planes showed that an
error of about 5 per cent. would have been incurred
had they not been resolved, the error being less because
the aluminium line does not broaden towards the base as
much as the tungsten line does.

A further illustration of the magnitudes which such
errors can attain is given in fig. 4, which represents part
of a record of silver radiation filtered by a palladium
filter and scattered from a mixed layer of tungsten and
aluminium.   The record is explained by reference to
fig. 5, showing the positions of all the lines and steps which
occur in the photograph when silver K radiation is
reflected from a powder of tungsten and aluminium.

The arrows on the steps shown in fig. 5 indicate whether, going from right to left, the steps consist in an increase in the background, due to the sudden increase in film sensitivity at the silver absorption edge, or in a decrease, due to the absorption edge of palladium. The large number of steps and their great intensity have distorted the background very considerably from the smooth curve

Fig. 4.
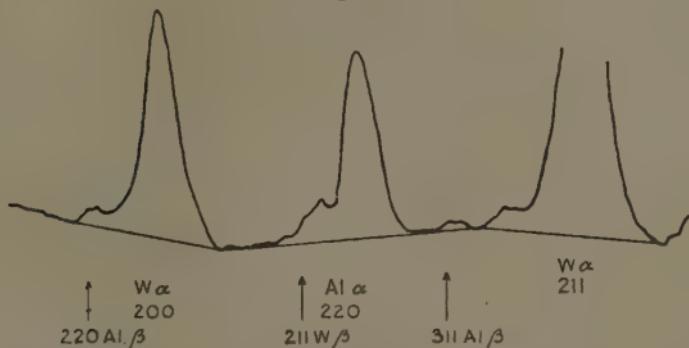
Record of a group of silver K reflexion lines filtered by a palladium filter. Example of badly deformed background due to discontinuities.

Fig. 5.
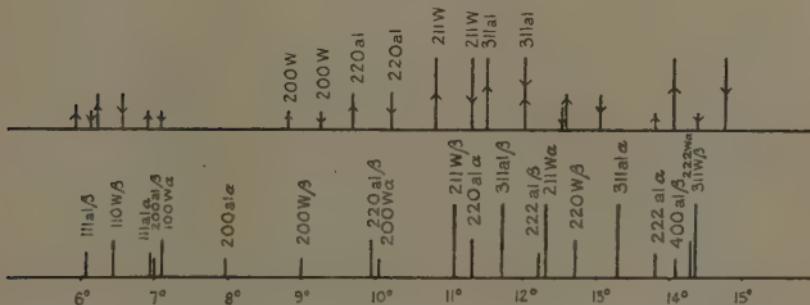
Diagram of lines (lower row) and background stops (upper row) for the reflexions of silver radiation, partially filtered by a palladium screen, reflected from a mixed layer of tungsten and aluminium. ↑ denotes the silver sensitivity edges, ↓ denotes the palladium absorption edges.

which it normally follows. The line most affected is the weak (200) tungsten K_α line, which lies between two steps caused by the stronger (220) reflexion from aluminium. The measured intensity of this line was found to be 20 per cent. different from the value which could be

ascribed to it with reasonable accuracy by extrapolation
from measurements on other lines less affected by steps,
and from measurements from other photographs in which
the silver steps were eliminated by the neutralizing
process discussed later in this paper.

Methods for avoiding Errors caused by Background Steps.

6. We have seen in § 4 that broad lines are not only
more exposed to interference from background steps,
but that errors incurred by such interference are greater.
The lines should therefore be as narrow as is convenient
for good measurement *. Their width depends on two
distinct factors—the size of the crystallites from which
reflexion takes place, and the resolving power of the
goniometer in the particular setting used. Extinction con-
siderations set an upper limit to the size of the crystallites,
for if the uncertain correction for the effects of extinction
is to be avoided, the crystal elements must be so small
that reduction of the reflected beam by extinction in the
individual crystallite is negligible. For light atoms such
as aluminium this corresponds to a thickness of about
1000 atom-planes ; for heavy elements such as gold the
thickness is only of the order of 100 atom-planes, and the
reflexion lines are then broadened by the limited resolving
power of the crystallites. No arrangement of the goni-
ometer can reduce this width, but the setting should be
such that no appreciable further broadening is introduced.
For the same reason the crystallites should not be appreci-
ably smaller than the maximum size permitted by the
condition of negligible extinction. We found that
mechanical grinding often produces particles comprising
a great variety of sizes, some being too large to fulfil the
condition of negligible extinction and others so small
or so distorted by the process of grinding that their
resolving power is greatly impaired, giving a very broad
contribution to the line, and making it fade gradually
into the background. Such powders, comprising what
we have called a "vanishing-size distribution," make

* In a paper by one of us (J. Brentano, Proc. Phys. Soc. xlvii. p. 932
(1935)) it was indicated that lines with a flat top and a steep drop on
both sides are the best for quantitative intensity measurements. In
the light of the present discussion such lines have the advantage that the
angular range in which superpositions may occur is strictly limited.
The line-width must be sufficiently restricted to avoid superpositions
of relevant lines and of background steps.

correct location of the base-line very difficult, and the
evaluation of intensities correspondingly uncertain.
Chemical precipitation often yields particles of more
uniform size, and can sometimes be controlled to give
particles of the most suitable size by adjustment of the
conditions of preparation. When grinding is necessary
we have found that more uniform particles are obtained
if to the powder is added several times its own volume of
pulverized fused silica. This gives no disturbing addi-
tional lines or background steps. The addition of silica
is also useful when a comparison substance is added to
the powder, so that the atomic scattering factors may be
related one to another without introducing any explicit
measurements of the absorption coefficient [*]. In such
measurements the two powders must consist of so fine
particles and be so thoroughly mixed that one single
particle or group of particles of one substance is not
sufficiently thick to absorb an appreciable part of the
radiation incident upon it [†]. If the particle size conforms
to the limit imposed by the condition of negligible extinc-
tion, this condition is usually adequate to ensure also
negligible absorption in one particle, and the crushing
of the powders by the hard silica seems to be singularly
effective in obtaining the required intimacy of mixing.

In order to take account of the effect of background steps
on the measurement of lines, the width of the reflexions,
which in the case of a mixed powder may be different for
different constituents, is obtained by direct measurement,
which offers a check on the sizes of the crystallites.
The width of the lines and steps is then used in a plan
drawn from calculation, such as is shown in fig. 5, in which
all the lines and background steps are inserted according
to their angular position and to their relative magnitude.
From this plan those lines can be selected which are free
from superposition with background steps. Frequently
it is found that only few lines are free from interference,
and that measurements have to comprise lines situated
close together or near to background steps. For instance,
in comparing tungsten and aluminium reflexions of silver
K-radiation, the case illustrated by fig. 5, we can see that
the (200) and the (311) K_a aluminium reflexions are
adequately isolated and free of superpositions, but

[*] J. Brentano, Phil. Mag. iv. p. 621 (1927).
[†] J. Brentano, Phys. Soc. Proc. xlvii. p. 932 (1935).

that the best-placed tungsten line is (211) K_α, and that its measurement is affected by the neighbourhood of the (222) K_β reflexion of aluminium, unless the K_β radiation is completely removed by a filter. In most of our work such thick filters were used as to practically suppress the K_β radiation. In this case, in order to avoid increasing the exposure unduly, only partial filtering was used, and the residual effect of the K_β reflexion on the evaluation of the (311) K_α reflexion of tungsten was calculated and allowed for from the measured intensity of a strong K_β line.

It will be seen that each additional reflexion and each additional filter introduces new discontinuities and decreases the chances of finding lines sufficiently separated from other lines or from background steps. Aluminium, like any substance constituted of light atoms, as a comparison substance has the advantage of having no absorption edges in the range of wave-length generally used, and so of introducing no additional steps. Its reflexions of existing discontinuities add, however, to the number of steps.

The conditions are less favourable with substances giving very crowded reflexion patterns or constituted of molecules with a great number of atoms having absorption edges in the wave-length range of the white radiation emitted by the bulb. And again, a powder mixture gives a more complex pattern than a powder of a single substance. This has some bearing on the method of mixed powders for comparative intensity measurements, a method developed by one of us with the object of eliminating absorption coefficients from the evaluation of intensities. A survey of the various aspects of intensity measurements and a comparison of the methods has been given in a short note *.

7. One way indicated in this note of avoiding the increased number of lines and steps resulting with a mixture is to use a substitution method in which the two substances to be compared are exposed alternately to X-rays and the reflexions recorded on separate films. The lines and steps associated with one substance are thus clear of those of the other. An arrangement by which this can be done has been described by one of us †. This substitution method

* J. Brentano, *Zeits. f. Phys.* xcix. p. 65 (1936).
† J. Brentano, Phys. Soc. Proc. xlix. p. 61 (1937).

is a convenient means of comparing the qualitative appearance of reflexions from two specimens when recorded under closely similar conditions, or of distinguishing the broadening of lines introduced by the goniometer from the broadening due to the smallness of the crystallites, by choosing for one of the specimens a powder constituted of large crystallites and giving good reflexions. Different preparations of the same substance can thus easily be compared. It should be noted, however, when examining substances with different absorption coefficients, that for quantitative intensity measurements this method requires that the absorption coefficients of the X-rays in each of the specimens be explicitly known. This is not easy to satisfy to the required accuracy. Referring to the particular instance of our dispersion measurements, the absorption coefficients should have been known for each of the various wave-lengths used, which could only have been found by special determinations using monochromatic radiation or on the basis of a theoretical treatment of dispersion, the checking of which was the aim of our experiments. Quite apart from our experiments, it is difficult to obtain sufficiently accurate absorption coefficients which can be applied to the aggregation of the powder with binding material. As the absorption law is exponential in character, and as the absorption coefficients according to the particular conditions enter the intensity expression in a linear or in an exponential way, deviations from the correct value have a very marked effect on the results. In this connexion it should be noticed that objections have been raised by Schäfer * and by Brindley and Spiers † against the mixed powder method on the ground

* C. Schäfer, *Zeits. f. Phys.* lxxxvi. p. 738 (1933).

† Brindley and Spiers, Proc. Leeds Phil. Lit. Soc. iii. p. 73 (1935), have advocated a substitution method independently of us, and advise it for quantitative measurements. They have arranged the powders so that the reflexions from the two specimens are recorded on the same film, thus giving the same superpositions as are obtained from a mixture of powders, but eliminating the need for intimate mixing of the powders. As we have said, the mixing process has not been a source of difficulty, and errors encountered by other observers seem to be caused by the use of too large particles, resulting in both extinction and absorption errors. A detailed discussion of these conditions has appeared in another paper (J. Brentano, *l. c.*). *Added in proof* : A letter informs us that Messrs. Brindley and Spiers have abandoned the substitution method in the form in which they introduced it first and have adopted a similar method as described by us for recording the reflexions from the specimens separately.

that a very intimate mixing of the powders is necessary
if the absorption term is to be eliminated. Using the
technique of admixing a brittle substance like fused silica,
as already described, we found no difficulty in satisfying
this condition. Whether the mixture is sufficiently fine
can easily be checked by comparing the relative intensities
of the reflexions from the constituents of the mixed
powder for two wave-lengths so chosen that any absorption
effect (which varies with wave-length) plays a widely
different part for the two wave-lengths selected, and
also chosen so as to be remote from any large anomalous
dispersion effect : an absorption error due to the particles
of one of the substances is then shown by a change in the
relative intensities of the reflexions for the two wave-
lengths. No similarly sensitive check is offered to
ascertain that the absorption coefficients are correctly
assessed in those methods where they enter explicitly
in the calculation. It is for this reason that we preferred
the method of mixed powders for our dispersion measure-
ments, and that we consider that it gives the more reliable
results in all those cases where the required conditions
are satisfied. One requirement is a powder of sufficiently
small and sufficiently well mixed particles, and here the
major condition that the particles should be sufficiently
small is in general already imposed by the requirements
of negligible extinction. The other requirement is that
lines should be found which are not affected by super-
position with other lines or with background steps.

8. Another way of reducing the number of super-
positions—and this we adopted in our measurements with
mixed powders—is to consider more closely the various
steps with a view to eliminating some of them and so
reducing their number.

The steps may conveniently be divided in three
groups :—

(*a*) Steps caused by selective absorption of radiation
of a wave-length for which the sensitivity of the photo-
graphic material is similar to the sensitivity for the
characteristic radiation of which the reflexions are
measured.

(*b*) Selective absorption steps like those in (*a*), which lie,
however, in a range of wave-lengths for which the photo-
graphic emulsion has a higher sensitivity than for the

characteristic radiation of the reflexion lines. This is found when the step occurs for a wave-length somewhat shorter than the wave-length of the silver or bromine K absorption edge, while the wave-length of the characteristic radiation is somewhat longer. In relation to the monochromatic reflexion lines such steps may be four to five times greater than the steps of type (*a*).

(*c*) Steps caused by the sudden change in film sensitivity at the absorption edges of silver and bromine. These steps are of the same order of magnitude as the steps of type (*b*).

The measured, just separated steps of nickel we discussed in (5) with regard to the magnitude of the possible errors are examples of steps of type (*a*), and we see, therefore, that while such steps when unresolved lead to errors which will only affect measurements of high accuracy, steps of types (*b*) and (*c*) may introduce errors of the order of 20 per cent., and may, therefore, affect even measurements of moderate accuracy.

We have developed the use of a neutralizing filter system which will eliminate some of these major steps. It can eliminate those of type (*c*), and often reduces those of type (*b*) sufficiently not to be disturbing. The incident beam is passed through a filter of silver or bromine of such a thickness that the ratio of the intensities of white radiation transmitted through the filter on the two sides of the silver or bromine absorption edge is reciprocal to the ratio of the sensitivities of the photographic material at the two sides of the edge. The photographic action of white radiation is thus made equal on the two sides of the silver or bromine absorption or sensitivity edge, and steps of type (*c*) are eliminated. For the particular photographic film used the thickness of the silver screen was 0·002 cm. ; the bromide screen, made of potassium bromide in Canada balsam, was adjusted by trial to the correct thickness.

The neutralizing process is not of universal applicability, nor is this in all cases necessary. The absorption coefficient increases by about eight times as we pass across the K absorption edge to shorter wave-lengths, and changes on each side according to a law in which the wave-length appears approximately as $(\lambda/\lambda_0)^3$, where λ is the wave-length of the particular radiation considered and λ_0 the wave-length of the absorption edge. We see,

therefore, that for wave-lengths $\lambda > 2\lambda_0$ the absorption coefficient is greater than the peak-value which it reaches at the absorption edge, while for wave-lengths $\lambda < \frac{1}{2}\lambda_0$ the absorption coefficient is less than its minimum value, which occurs just to the long-wave side of the edge. A filter giving selective filtering for the edge λ_0 will therefore be practically totally absorbing for $\lambda > 2\lambda_0$, and will lose its selective properties for $\lambda < \frac{1}{2}\lambda_0$. Applying this in examining the use of the silver or bromine filter, it is convenient to distinguish between the following cases in which λ_c is the wave-length of the characteristic radiation and λ_0 the silver or bromine absorption edge.

If $\lambda_c > 2\lambda_0$, the absorption of the characteristic radiation by the neutralizing silver or bromine filter will be very large, so that the filter cannot be introduced without considerable intensity loss. In this case it is better to reduce the potential difference applied to the X-ray tube to such a value that practically no white radiation of shorter wave-lengths than λ_0 is emitted. This will limit the output of characteristic radiation, but in view of λ_c being so much greater than λ_0, the potential applied at the bulb can be considerably in excess of the minimum potential for exciting λ_0. The disadvantage of a longer exposure time is amply offset by the greatly reduced background when hard white radiation is absent. To obtain the best X-ray output at the reduced voltage imposed by the condition of avoiding white radiation of a wave-length shorter than λ_0, we found a great advantage in using a rectified smoothed high-tension supply giving practically constant voltage, with which the potential difference at the X-ray tube can be maintained near the optimum limit.

If $2\lambda_0 > \lambda_c > \lambda_0$, the process of neutralizing the unequal sensitivity of the photographic material by a filter must be used, for the potential applied at the bulb could not be reduced sufficiently without impairing too much the output of characteristic radiation. In certain circumstances, however, the filter used to remove the K_β and K_γ components of the characteristic radiation can be arranged so as to remove also the white radiation for the wave-length λ_0; an additional neutralizing filter then becomes unnecessary.

If $\lambda_0 > \lambda_c > \frac{1}{2}\lambda_0$, there is no alternative to neutralization if the sensitivity step is to be removed. The exposure-

time is increased by the neutralizing filter because the
advantage that the characteristic radiation lies now in
the region of high sensitivity is lost by the action of the
neutralizing filter.

If $\lambda_c < \frac{1}{2}\lambda_0$, no interference from the white radiation in
the region λ_0 should arise, because the filter used for
removing the K_β and the K_γ components of the character-
istic radiation will also remove the white radiation in the
region of λ_0.

It is difficult to estimate by calculation the magnitude
of the various discontinuities appearing in the white
radiation as each absorber, apart from introducing its
own discontinuities, modifies the intensity distribution
of white radiation for all wave-lengths, and so alters the
discontinuities produced by other absorbers. A con-
venient way which we have adopted is to analyse the
radiation by means of a single rotating crystal chosen so as
to have no selective properties. Not only is the effect
of each filter immediately indicated, and the relative
magnitudes of the discontinuities measured, but also any
residual K_β or K_γ components of the characteristic radia-
tion are shown, and their intensities measured in terms
of that of the K_α radiation. Due allowance can then
be made for such faint K_β or K_γ lines if they fall too close
to a K_α line to be resolved.

9. Apart from the two procedures here discussed—the
substitution method and the avoidance of superpositions
by selection of suitable lines and by elimination of some
absorption steps—the possibility exists of eliminating or
greatly reducing the white radiation which gives origin
to the discontinuous background by using a single crystal
which acts as a monochromator and allows only mono-
chromatic radiation surrounded by a narrow band of
white radiation, plus the incoherently scattered radiation
from the single crystal, to enter the goniometer. We
found it unnecessary to recur to this procedure, which
on the one hand does not fully eliminate white radia-
tion, and on the other greatly restricts the angular width
of the beams and increases the times of exposure.

10. It should be emphasized that the discontinuities
in the background discussed here have little or no sig-
nificance in the routine determinations of crystal structures

by angular measurements, in which intensities need only be known approximately. They are relevant in those cases in which the comparison of relative intensities of lines is significant, when it appears necessary to examine the background in the neighbourhood of the lines and to take account of any discontinuities in their evaluation. The point we wish particularly to stress is that accuracy in quantitative intensity measurements is not attained by averaging over a greater number of reflexions, without considering the systematic errors introduced by intensity steps of the background, but rather by making an analysis of the location of these steps and selecting the lines free from interference.

XLVIII. *On Boole's Operators in the Calculus of Finite Differences.* By H. FLORIN, *D.Phil., Louvain* *.

1. A GENERALIZATION of Boole's operators Π and ρ [1] is due to Milne-Thomson [2], who has made extensive applications of these operators.

We define these operators as follows:

$$\Pi u(x) = x\{u(x) - u(x-1)\} \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\rho^m u(x) = \frac{\Gamma(x+1)}{\Gamma(x-m+1)} u(x-m), \quad . \quad . \quad . \quad (2)$$

where m can be complex.

These definitions are somewhat less general than Milne-Thomson's in order to simplify our formulas without losing the essential properties.

The object of the present paper is to prove that it is possible, by means of an integral transformation, to consider Π as an ordinary variable. This integral transformation is of the same kind as Bromwich's integral

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interpretation of the operator p in Heaviside's theory [3]. This process enables us to extend the theory of the Π-operator, especially in the case when this operator occurs in the denominator. It will also be possible to express the operator ρ in Π.

In §3 of this paper we shall consider another integral transformation, which changes ρ into an ordinary variable, and finally we shall discuss the relations of the investigated transformations with transformations usual in other operational methods.

2. We can consider Π as an ordinary variable by means of the transformation

$$u(x) = \frac{\Gamma(x+1)}{2\pi i} \int_C \frac{\phi(\Pi)\, d\Pi}{\Pi\Gamma(x-\Pi+1)}, \quad . \quad . \quad . \quad (3)$$

where C is a contour including all poles in the finite part of the plane and ϕ is assumed to possess no other singularities in the finite part of the plane than a finite number of poles.

To verify this, we can write by (1)

$$\Pi u(x) = x\{u(x)-u(x-1)\} = \frac{x\,\Gamma(x-1)}{2\pi i} \int_C \frac{\phi(\Pi)\, d\Pi}{\Pi\Gamma(x-\Pi+1)}$$

$$\div \frac{\Gamma(x-1)}{2\pi i} \int_C \frac{\phi(\Pi)(\Pi-x)\, d\Pi}{\Pi\Gamma(x-\Pi+1)} = \frac{\Gamma(x-1)}{2\pi i} \int_C \frac{\Pi\phi(\Pi)\, d\Pi}{\Pi\Gamma(x-\Pi+1)}.$$

$$. \quad . \quad . \quad (4)$$

The symbol Π, which is an operator in the left part of the equation, must be interpreted as an ordinary variable in the following parts.

We shall indicate the relation between $u(x)$ and $\phi(\Pi)$ by the sign

$$u(x) \doteqdot \phi(\Pi). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus, by (4),

$$\Pi u(x) \doteqdot \Pi\phi(\Pi), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and in particular, since

$$\frac{\Gamma(x+1)}{2\pi i} \int_C \frac{d\Pi}{\Pi\Gamma(x-\Pi+1)} = 1, \quad . \quad . \quad . \quad (7)$$

we have

$$1 \doteqdot 1; \quad \Pi 1 \doteqdot \Pi. \quad . \quad . \quad . \quad . \quad (8)$$

Obviously the left side of the last form equals zero.

The Operator ρ.

Applying the same transformation (3) to the operator ρ, we get by (2)

$$\rho u(x) = xu(x-1) = \frac{x\Gamma(x)}{2\pi i} \int_C \frac{\phi(\Pi)\,d\Pi}{\Pi\Gamma(x-\Pi)}. \quad . \quad (9)$$

By changing the variable Π into $\Pi-1$ this becomes

$$\rho u(x) = \frac{\Gamma(x+1)}{2\pi i} \int_C \frac{\phi(\Pi-1)\,d\Pi}{(\Pi-1)\Gamma(x-\Pi+1)};$$

if C is taken sufficiently large, this contour can remain unchanged by Cauchy's theorem.

Thus we have, by (3),

$$\rho u(x) \doteqdot \frac{\Pi}{\Pi-1}\phi(\Pi-1), \quad . \quad . \quad . \quad (10)$$

and in particular

$$\rho 1 \doteqdot \frac{\Pi}{\Pi-1}. \quad . \quad . \quad . \quad . \quad (11)$$

Similarly we can deduce for arbitrary finite real or complex m

$$\rho^m u(x) \doteqdot \frac{\Pi}{\Pi-m}\phi(\Pi-m). \quad . \quad . \quad (12)$$

Already with these results it is possible to deduce Boole's and Milne-Thomson's theorems upon ρ and Π in a simplified way.

As an example we consider the following theorem :—

If $F(\lambda)$ is a polynomial, we have

$$F(\Pi)\rho^m u(x) = \rho^m F(\Pi+m)u(x) \quad . \quad . \quad . \quad (13)$$

and

$$\rho^m F(\Pi)u(x) = F(\Pi-m)\rho^m u(x). \quad . \quad . \quad (14)$$

In our method the restriction " polynomial " for $F(\lambda)$ can be replaced by the less stringent restrictions mentioned below (3), and we obtain besides (13) and (14), from (12) :

$$F(\Pi)\rho^m u(x) \doteqdot F(\Pi)\rho^m \phi(\Pi) = \frac{\Pi}{\Pi-m} F(\Pi)\phi(\Pi-m) \quad (15)$$

and

$$\rho'''\mathrm{F}(\varPi)u(x) \doteq \rho'''\mathrm{F}(\varPi)\phi(\varPi) = \frac{\varPi}{\varPi-m}\,\mathrm{F}(\varPi-m)\phi(\varPi-m),$$

$$\cdots \quad (16)$$

where $u(x)\doteq\phi(\varPi)$.

Our form (12) is of particular interest in the special case

$$\rho'''1 \doteq \frac{\varPi}{\varPi-m}\,\cdots\quad\cdots\quad (17)$$

By using the residue theorem we have by (3)

$$\frac{\varGamma(x+1)}{\varGamma(x-m+1)} \doteq \frac{\varPi}{\varPi-m}, \quad \cdots \quad (18)$$

in agreement with definition (2) of ρ^m.

It is easy to express x as a function of $\varPi$, for from (1) and (2) we get

$$xu(x) = (\varPi+\rho)u(x), \quad \cdots \quad (19)$$

and by (10)

$$xu(x) \doteq \left(\varPi + \frac{\varPi}{\varPi-1}\,\varTheta^{-1}\right)\phi(\varPi) \quad \cdots \quad (20)$$

(where $\varTheta^{-1}\phi(\varPi)=\phi(\varPi-1)$), and, if $u(x)$ is a polynomial,

$$u(x) \doteq u\left(\varPi + \frac{\varPi}{\varPi-1}\,\varTheta^{-1}\right)1. \quad \cdots \quad (20\,a)$$

Taking in (3) $\phi(\varPi)=\varPi^s$, we have for s integer, and neither negative nor zero,

$$0 \doteq \varPi^s, \quad \cdots \quad \cdots \quad (21)$$

since the integrand has no poles.

Hence, putting $u(x)=1$, we obtain, using (20) and (21),

$$x \doteq \frac{\varPi}{\varPi-1}. \quad \cdots \quad \cdots \quad (22)$$

This is a special case of (18):

$$\frac{\varGamma(x+1)}{\varGamma(x-s+1)} \doteq \frac{\varPi}{\varPi-s}.$$

Our results give very easily the solution of some difference equations.

So we consider difference equations of the form

$$F(\Pi)u(x) - G(x) = 0, \quad \ldots \quad (23)$$

where F is a polynomial and $G(x)$ is a function which can be written as a function of Π, *e. g.*, a polynomial in x. In the last case $G(x)$ can be expressed as a function of Π by means of (20) and (22); otherwise (18) can often be used.

Applying the transformation (3) to (23), we obtain

$$-\frac{\Gamma(x+1)}{2\pi i} \int_0 \frac{F(\Pi)\phi(\Pi) - \psi(\Pi)}{\Pi\Gamma(x-\Pi+1)} d\Pi, \quad \ldots \quad (24)$$

where

$$u(x) \doteqdot \phi(\Pi); \quad G(x) \doteqdot \psi(\Pi). \quad \ldots \quad (25)$$

Now we must determine $\phi(\Pi)$ in such a way that the integral vanishes. For this purpose it suffices that the integrand has no poles (save at infinity). Hence we can write

$$F(\Pi)\phi(\Pi) - \psi(\Pi) = \sum_{k=1}^{n} c_k \Pi^k, \quad \ldots \quad (26)$$

where the c_k are arbitrary constants and n is arbitrary too; but for our purpose, we can put n equal to the order of the difference equation in order to obtain a general solution of our problem.

From (26) we get

$$\phi(\Pi) = \frac{\chi(\Pi)}{f(\Pi)}. \quad \ldots \quad \ldots \quad (27)$$

An easy way to interpret this result can be given in the form of a " partial-fraction rule " [4].

By division it is always possible to obtain a numerator $\bar{\chi}$ of the same or lower degree than $f(\Pi)$ if χ and f are polynomials :

$$u(x) \doteqdot \phi(\Pi) = \frac{\chi(\Pi)}{f(\Pi)} = \frac{\bar{\chi}(\Pi)}{f(\Pi)} + A_1\Pi + A_2\Pi^2 + \ldots + A_\alpha\Pi^\alpha.$$

The terms

$$A_k\Pi^k \quad (k > 0, \text{ integer})$$

vanish in our interpretation by (21).

When all solutions a of $f(\lambda) = 0$ are different, and none of them is zero, we can write

$$u(x) \doteq \frac{\overline{\chi}(\Pi)}{f(\Pi)} = \frac{\overline{\chi}(0)}{f(0)} + \sum_a \frac{\overline{\chi}(a)}{af'(a)} \frac{\Pi}{\Pi - a} \quad . \quad . \quad (28)$$

Hence, by (8) and (18), we have

$$u(x) \doteq \frac{\overline{\chi}(0)}{f(0)} + \sum_a \frac{\overline{\chi}(a)}{af'(a)} \frac{\Gamma(x+1)}{\Gamma(x-a+1)}. \quad . \quad . \quad (29)$$

If some a's are equal, or zero, our expression will contain not only these terms but also

$$\frac{k_s \Pi}{(\Pi - a)^s}, \qquad \begin{array}{l} s > 1 \text{ ; integer} \\ k_s = \text{constant} \end{array} \bigg\}, \quad . \quad . \quad (30)$$

where a may be zero.

We interpret these terms by (3), *e. g.*, when $s = 2$ we have

$$\frac{\Pi}{(\Pi - a)^2} \doteq -\frac{\Gamma(x+1)}{\Gamma(x-a+1)} \Psi(x-a+1), . \quad . \quad (31)$$

where $\Psi(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda)$.

In more complicated cases we can expand $\phi(\Pi)$ in rational fractions by Mittag Leffler's theorem.

The general solution of the difference equation is obtained by substituting the constants c_k (*cf.* (26)) in the definitive solution by arbitrary periodic functions $p_k(x)$ of period unity.

Milne-Thomson has stated the possibility of expressing a factorial series by means of the operator ρ :

$$\sum_{s=0}^{n} a_s \rho^{m-s} 1 = \frac{\Gamma(x+1)}{\Gamma(x-m)} \sum_{s=0}^{n} \frac{a_s}{(x-m)(x-m+1) \ldots (x-m+s)}.$$

By (18) we can write this

$$\frac{\Gamma(x+1)}{\Gamma(x-m)} \sum_{s=0}^{n} \frac{a_s}{(x-m)(x-m+1) \ldots (x-m+s)} \doteq \sum_{s=0}^{n} \frac{a_s \Pi}{\Pi - m + s}.$$

$$. \quad . \quad . \quad (32)$$

If the series converge, or can be made convergent by additional terms of the form (32), n may tend to infinity.

3. In the foregoing we have used a transformation changing the operator Π into a variable ; ρ remained an

operator. Now we shall consider a transformation changing the operator ρ into an ordinary variable ρ. For this purpose we put

$$u(x) = \frac{\Gamma(x+1)}{2\pi i} \int_L \rho^{-x-1} e^\rho \phi(\rho) \, d\rho, \quad . \quad . \quad (33)$$

where L is a contour integral starting at $-\infty$, encircling the origin clockwise, and returning to $-\infty$; $\phi(\rho)$ is assumed to be such a function that the integral exists; we mean the branch of ρ^{-x-1} satisfying the inequality

$$-\pi \leqq \arg \rho \leqq \pi.$$

Then we have

$$\rho u(x) = x u(x-1) = \frac{x\Gamma(x)}{2\pi i} \int_L \rho^{-x-1} e^\rho \rho \phi(\rho) \, d\rho,$$

or

$$\rho u(x) \doteqdot \rho \phi(\rho), \quad . \quad . \quad . \quad . \quad (34)$$

and since

$$\frac{1}{\Gamma(x+1)} = \frac{1}{2\pi i} \int_L \rho^{-x-1} e^\rho \, d\rho, \quad . \quad . \quad (35)$$

we have

$$1 \doteqdot 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The operator Π is transformed in the following way (provided that $\lim\limits_{\rho \to -\infty} \rho^{-x} e^\rho \phi(\rho) = 0$):

$$\Pi u(x) \doteqdot \rho \frac{d}{d\rho} \phi(\rho). \quad . \quad . \quad . \quad . \quad (37)$$

This can be verified by (1) and (33):

$$\Pi u(x) = x\{u(x) - u(x-1)\} = \frac{x\Gamma(x+1)}{2\pi i} \int_L \rho^{-x-1} e^\rho \phi(\rho) \, d\rho$$
$$- \frac{x\Gamma(x)}{2\pi i} \int_L \rho^{-x} e^\rho \phi(\rho) \, d\rho.$$

When we integrate by parts the first term, since the integrated terms vanish, we have

$$\Pi u(x) = \frac{\Gamma(x+1)}{2\pi i} \int_L \rho^{-x} e^\rho \{\phi(\rho) + \phi'(\rho)\} \, d\rho$$
$$- \frac{\Gamma(x+1)}{2\pi i} \int_L \rho^{-x} e^\rho \phi(\rho) d\rho = \frac{\Gamma(x+1)}{2\pi i} \int_L \rho^{-x-1} e^\rho \rho \phi'(\rho) \, d\rho.$$

Since

$$x = \rho + \Pi,$$

we have

$$xu(x) \doteq \rho \left(1 + \frac{d}{d\rho}\right) \phi(\rho), \quad . \quad . \quad . \quad (38)$$

and when u is a polynomial

$$u(x) \doteq u\left(\rho + \rho \frac{d}{d\rho}\right) 1. \quad . \quad . \quad . \quad (38\,a)$$

In many cases a difference equation will be transformed by (37) and (38) into a differential equation.

Already Boole had mentioned the analogy of the operators Π and ρ with d/dx and $\exp x$. We can now formulate this analogy by executing the further transformation

$$\rho = e^s. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

For let

$$u(x) \doteq \phi(\rho) \doteq \chi(s); \quad . \quad . \quad . \quad . \quad (40)$$

then we get

$$\rho u(x) \doteq \rho \phi(\rho) \doteq e^s \chi(s) \quad . \quad . \quad . \quad . \quad (41$$

and

$$\Pi u(x) \doteq \rho \frac{d}{d\rho} \phi(\rho) \doteq \frac{d}{ds} \chi(s). \quad . \quad . \quad (42)$$

We shall not investigate the applications of the theory developed in this paper.

4. It is of some interest to discuss the connexions between the transformations pointed out in this paper and their relations to other transformations of the same kind.

The transformations with which we have been dealing must satisfy the following condition :—

We want an integral transformation,

$$u(x) = \int_L K(x, \xi) \phi(\xi) \, d\xi, \quad . \quad . \quad . \quad (43)$$

transforming a linear operator into another linear operator, in most cases into a simple multiplication.

So in the case of transformation (3) we wanted to get an integral transformation transforming the operator Π into a multiplicative operation,

$$\Pi u(x) \doteq \xi \phi(\xi) \quad . \quad . \quad . \quad . \quad . \quad (44)$$

or

$$\Pi u(x) = \int_{\mathrm{L}} \mathrm{K}(x,\,\xi)\xi\phi(\xi)\,d\xi.$$

By (1) we get

$$xu(x) - xu(x-1) = \int_{\mathrm{L}} \mathrm{K}(x,\,\xi)\xi\phi(\xi)\,d\xi. \qquad . \quad (45)$$

We can satisfy (45) when

$$x\mathrm{K}(x,\,\xi) - x\mathrm{K}(x-1,\,\xi) = \xi\mathrm{K}(x,\,\xi).$$

The solution of this difference equation is

$$\mathrm{K}(x,\,\xi) = \frac{\Gamma(x+1)}{\Gamma(x-\xi+1)}\mathrm{F}(x,\,\xi),$$

where $\mathrm{F}(x,\,\xi)$ is an arbitrary function of ξ and a periodic function of x of period 1.

The transformation must satisfy the further condition

$$1 \doteqdot 1$$

or

$$1 = \int_{\mathrm{L}} \mathrm{K}(x,\,\xi)\,d\xi.$$

We can satisfy this relation by putting

$$\mathrm{F}(x,\,\xi) = \frac{1}{2\pi i\xi},$$

and taking L equal to an appropriate contour. So we can get (3).

The integral transformation (33) is obtained by replacing the condition (44) by

$$\rho u(x) \doteqdot \xi\phi(\xi), \quad . \quad . \quad . \quad . \quad (49)$$

and, moreover, as in the former case, the condition

$$1 \doteqdot 1.$$

It is possible to consider Bromwich's integral transformation [5] for the Heaviside operator p from this point of view. Instead of (44) we want then the condition

$$\frac{d}{dx}u(x) \doteqdot \xi\phi(\xi), \quad . \quad . \quad . \quad . \quad (50)$$

which is verified by

$$\mathrm{K}(x,\,\xi) = e^{x\xi}\mathrm{F}(\xi), \quad . \quad . \quad . \quad . \quad (51)$$

where F is arbitrary. In order to satisfy

$$1 \doteq 1$$

we can take

$$u(x) = \frac{1}{2\pi i} \int_C \frac{e^{x\xi}}{\xi} \phi(\xi) \, d\xi, \quad . \quad . \quad . \quad (52)$$

C being (in simple cases) a large circle. This is Bromwich's integral.

Another very useful integral in the modern Heaviside operational calculus is Carson's integral [6],

$$u(x) = x \int_0^\infty e^{-x\xi} \phi(\xi) \, d\xi. \quad . \quad . \quad . \quad (53)$$

We can consider this integral from our point of view by inverting (50),

$$x u(x) \doteq \frac{d}{d\xi} \phi(\xi)$$

or

$$x u(x) = \int_L K(x, \xi) \frac{d}{d\xi} \phi(\xi) \, d\xi. \quad . \quad . \quad (54)$$

By integration by parts we obtain, when the integrated term vanishes,

$$x K(x, \xi) = - \frac{\partial K(x, \xi)}{\partial \xi}; \quad K(x, \xi) = e^{-x\xi} F(x) \quad \text{(F arbitrary)}.$$

We take the path of integration L and the function F in such a way that $1 \doteq 1$ and that the integrated term vanishes.

All these conditions are satisfied if $\phi(0) = 0$ by taking the positive real axis as path of integration and $F = x$, provided that the integrals (53) and (54) converge. So we get Carson's formula (53). It is easy to extend this reasoning to the case $\phi(0) \neq 0$, using Van der Pol's results [7].

Summary.

Boole's operators (in Milne-Thomson's generalized form) Π and ρ are treated as ordinary variables by means of integral transformations (3) and (33).

Some relations are obtained, *e. g.*, (17), (18), (19), (20), (20 *a*), (37), (38), (38 *a*).

The connexions between the transformations (3) and (33) and some similar integral transformations due to Bromwich and Carson are examined, and a heuristic method to find such integral transformations is described.

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April 1937,

XLIX. *Laplacian Transform Circuit Analysis.* By LOUIS A. PIPES, *Electrical Engineering Department, The Rice Institute, Houston, Texas* *.

Introduction.

IT is the purpose of this paper to present a method for the general solution of linear lumped circuits with constant coefficients by a general operational process. The method will first be developed for a general series circuit, and then, by the use of matrix notation, it will be generalized for the case of the most general *n*-mesh circuit.

The method here presented is adequate for the solution of a great many mechanical problems whose boundary conditions may all be expressed at some definite time, which, by a transformation, may be placed equal to $t=0$.

The Laplacian Transformation.

In a note in the Philosophical Magazine, vii. p. 1153 (1929), Van der Pol showed how Heaviside's methods might be modified to include more general boundary conditions. The method which Van der Pol suggested is based on the Laplacian transformation.

The application of this method to circuit analysis when expressed in matrix notation results in a general method which yields both the steady state and transient conditions of any constant linear lumped network.

To introduce the subject in a manner which will seem natural to those versed in circuit theory, let us consider the general lumped linear series circuit which has a general electromotive force $E(t)$ impressed upon it at $t = 0$.

The general equation of this circuit, which must hold at any time $t > 0$, is

$$L\, di/dt + Ri + Sq = E(t), \quad . \quad . \quad . \quad . \quad (1)$$

where
$\quad$ L = Coefficient of inductance.

$\quad$ R = Coefficient of resistance.

$\quad$ S = Coefficient of elastance.

$\quad$ i = the current in the circuit.

$\quad$ q = the charge on the condenser.

Since q represents the charge on the condenser, it may be considered as defined by

$$q(t) = \int_{-\infty}^{t} i\, dt. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We may, therefore, write our equation in the form

$$L\frac{di}{dt} + Ri + S\int_{-\infty}^{t} i\, dt = E(t). \quad . \quad . \quad . \quad (3)$$

Let us suppose that at $t = 0$ we have the boundary conditions

$$i = i_0, \qquad q = q_0, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, our circuit has an initial charge and an initial current.

Following Van der Pol, let us multiply our equation through by e^{-pt} and integrate from 0 to infinity. Here p is a complex number whose real part is greater than 0.

On doing this, we have

$$L\int_{0}^{\infty} \frac{di}{dt} e^{-pt} dt + R\int_{0}^{\infty} e^{-pt} i\, dt + S\int_{0}^{\infty} e^{-pt} q\, dt = \int_{0}^{\infty} e^{-pt} E(t)\, dt.$$
$$. \quad . \quad . \quad (5)$$

Let us call $g(p) = \int_0^\infty e^{-pt} i \, dt.$

For brevity, let us introduce an integral operator Q defined by the equation

$$Q = \int_0^\infty e^{-pt} (\quad) \, dt.$$

We may then write

$$g(p) = Qi. \quad . \quad . \quad . \quad . \quad (6)$$

Let us also introduce

$$G(p) = QE(t). \quad . \quad . \quad . \quad . \quad (7)$$

Now by integration by parts we obtain

$$Q \, di/dt = [e^{-pt}i]_0^\infty + pQi$$
$$= -i_0 + pg(p), \quad . \quad . \quad . \quad . \quad (8)$$

where use has been made of the fact that the integrated term vanishes at the upper limit and $i = i_0$ at $t = 0$ the lower limit.

Also integrating by parts, we obtain

$$Qq = -1/p e^{-pt} q(t)]_0^\infty + 1/p Q \, dq/dt, \quad . \quad . \quad (9)$$

but $\qquad\qquad dq/dt = i \quad$ and $\quad q(0) = q_0,$

hence

$$Qq = 1/p q_0 + 1/p \, g(p). \quad . \quad . \quad . \quad (10)$$

Hence our integrated equation becomes

$$(Lp + R + S/p)g = G + Li_0 - (S/p)q_0. \quad . \quad . \quad (11)$$

Let us write

$$Z(p) = Lp + R + S/p, \quad . \quad . \quad . \quad . \quad (12)$$

then

$$g(p) = 1/Z(p)[G + Li_0 - (S/p)q_0] \quad . \quad (13)$$

and the current is related to the transform $g(p)$ by the integral equation

$$Qi = g(p). \quad . \quad . \quad . \quad . \quad (14)$$

Now since this is an integral equation of the Laplace type, its solution is well known and is obtained by means of the Fourier Integral. (See H. W. March, Am. Math. Society Bull. v. 33, p. 311 (1927).)

The solution of (14) is

$$i(t) = \frac{1}{2\pi j} \int_{-j\infty + c}^{+j\infty + c} g(p)e^{pt}\,dp, \quad \begin{array}{l} c > 0, \\ j = \sqrt{-1}. \end{array} \quad (15)$$

We have here $i(t)$ given explicitly as a line integral in the complex p plane. The constant c is chosen so that the path of integration will be entirely to the right of the singularities of $g(p)$. In the cases we are considering in this paper, $g(p)$ has pole singularities all lying to the left of the axis of imaginaries except for pole singularities at the origin.

In these cases it is possible to enclose all the singularities of the integrand by a large circle of radius R and evaluate the integral by summing the residues.

Fig. 1.
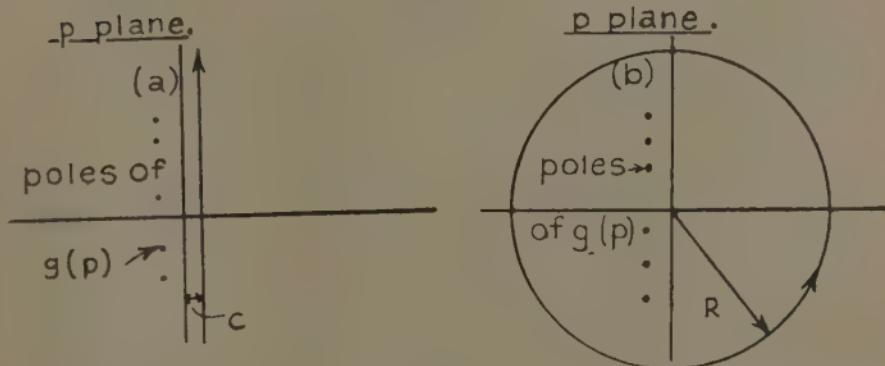
From the form of the answer, we see that $i(t)$ is equal to the sum of the residues of the function $g(p)e^{pt}$ which may be enclosed by the circle of radius R.

The preceding diagram (fig. 1) is helpful in this connexion.

The contour (*a*) may be distorted into the contour (*b*) which is a circle of radius R, which is large enough to include inside it all the singularities of the function $g(p)$.

In the case of lumped dissipative systems there are a finite number of poles all of which lie either on, or to the left of, the axis of imaginaries.

From a consideration of the distorted contour *b*, we may write explicitly

$$i(t) = \text{sum of residues of } e^{pt}g(p) \text{ in R.} \quad . \quad (16)$$

Since the poles of $g(p)$ are either simple poles or at most poles of low order, the calculation of the residues is a simple matter.

If in particular we have an impressed e.m.f. of the sinusoidal form $E(t) = E_m \sin wt$, we obtain

$$G(p) = QE(t) = E_m w/(p^2 + w^2)$$

by direct integration. In this case $g(p)$ becomes

$$g(p) = 1/Z(p) \left[\frac{E_m w}{p^2 + w^2} + Li_0 - \frac{Sq_0}{p} \right]. \quad . \quad . \quad (18)$$

It is seen that the poles of $g(p)$ depend essentially on the zeros of $Z(p)$. In the case that the roots of $Z(p) - 0$ are indistinct we have simple poles and the residues of $e^{pt}g(p)$ are most easily calculated. In the case $E(t) = 0$, our equation gives the free oscillations of the system subject to the initial charge and current boundary conditions. In any case we notice a simplification over the classical solution in that we have our arbitrary constants already evaluated without further labour.

The case of a constant e.m.f. impressed at $t = 0$ gives us $G(p) = QE = E/p$ and the solution is extremely simple.

The advantages of this method are more apparent, however, in the consideration of more complicated n mesh circuits. It is to a consideration of these circuits that we now turn. To discuss this matter more briefly we shall use matrix notation, and it will be necessary for us to make a short digression to consider the integration of a matrix.

Integration of a Matrix.

Before passing from the above simple case to the general case it is well to review briefly the definition of the integration of a matrix.

Let $[u]$ be a matrix whose elements are functions of the variable t. That is, $u_{ij} = u_{ij}(t)$ where $[u]$ is defined as

$$[u] = \left\| \begin{array}{c} u_{11} \ldots \ldots u_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_{n1} \ldots \ldots u_{nn} \end{array} \right\| . \quad . \quad . \quad (20)$$

Now if Q stands for our integral operator defined by

$Q = \int_0^\infty (\ )e^{-pt}\,dt$, we define as the integral $Q[u]$ of the matrix $[u]$ as being $[Qu]$ or that is

$$Q[u] = [Qu],$$

or by definition, the integral of a matrix is a matrix whose elements are the integrals of the original matrix.

Now let us suppose that $Qu_{ij} = g_{ij}(p)$. Then

$$Q[u] = [g(p)]. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Now for brevity we define P as being the integral operator

$$P = \frac{1}{2\pi j} \int_{-j\infty + c}^{+j\infty + c} e^{pt}(\ )\,dp. \quad . \quad . \quad . \quad (22)$$

We have the fundamental result of the Laplacian transform that if $Qu_{ik}(t) = g_{ik}(p)$, then

$$u(t) = P g_{ik}(p), \quad . \quad . \quad . \quad . \quad (23)$$

where c in the line integral must be chosen so that the path of integration is to the right of all the singularities of the $g_{ik}(p)$ functions.

We may therefore say that if $Q[u(t)] = [g(p)]$, then

$$[u(t)] = P[g(p)]. \quad . \quad . \quad . \quad (24)$$

This is merely a generalization expressed in matrix notation of the Laplacian transformation of a set of functions. We shall find this extremely useful in our subsequent operations. (24) may be considered as a generalization of (14) and (15).

The General N-Mesh Circuit.

We shall find the above matrix results of great utility in discussing the general n-mesh circuit.

The well-known canonical equations of such a network are

$$Z_{11}i_1 + \ldots \ldots + Z_{1n} = e_1(t)$$
$$\dot{Z}_{n1}i_1 + \ldots \ldots + Z_{nn} = e_n(t),$$

where

$(i_1 \ldots i_n)$ are the n-mesh currents,
$(e_1 \ldots e_n)$ are the n-mesh voltages,

2 L 2

The Z_{ij} quantities are impedance operators of the form

$$Z_{ij} = L_{ij}\, d/dt + R_{ij} + S_{ij} \int_{-\infty}^{t} (\ )\, dt,$$

where

L_{ij} = the coefficients of inductance.

R_{ij} = the coefficients of resistance.

S_{ij} = the coefficients of elastance.

If we now introduce the following matrices,

$$[Z] = \begin{bmatrix} Z_{11} \ldots Z_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ Z_{n1} \ldots Z_{nn} \end{bmatrix} \quad [e] = \begin{bmatrix} e_1 \\ \cdot \\ e_n \end{bmatrix} \quad [i] = \begin{bmatrix} i_1 \\ \cdot \\ i_n \end{bmatrix},$$

$$\ldots \ (26)$$

we may write the set of above equations in the convenient matrix form

$$[Z][i] = [e]. \quad \ldots \ldots \quad (27)$$

This is a generalization of the simple equation (3).

Now let us multiply the matrix equation (27) by $e^{-pt}\, dt$ and integrate from 0 to infinity. In other words we operate on both sides of (27) with our operator Q.

We thus obtain

$$Q[Z](i) = Q[e] = [G(p)], \quad \text{say.} \quad \ldots \quad (28)$$

This is a generalization of (5).

The elements of the column matrix $[G(p)]$ are given by $G_r = Q e_r$.

Let us now introduce the quantities $g_r = Q i_r$ and the matrix $[g(p)]$ a columnar matrix whose elements are the g_r quantities.

That is, we introduce the matrix

$$[g(p)] = Q[i] \quad \ldots \ldots \ldots \quad (29)$$

a generalization of (6).

Let us assume that the initial values at $t=0$ of the charges and currents in the n meshes are respectively $(q_1^0 \ldots q_n^0$ and $(i_1^0 \ldots i_n^0)$. We now introduce the matrices

$$[i^0] = \begin{bmatrix} i_1^0 \\ \cdot \\ i_n^0 \end{bmatrix} \quad [q^0] = \begin{bmatrix} q_1^0 \\ \cdot \\ \cdot \\ q_n^0 \end{bmatrix} \quad [Z(p)] = \begin{bmatrix} Z_{11} \ldots Z_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ Z_{1n} \ldots Z_{nn} \end{bmatrix},$$

$$\ldots \quad (30)$$

where $Z_{ij}(p) = L_{ij}p + R_{ij} + S_{ij}/p,$

$$[L] = \begin{bmatrix} L_{11} \cdot \ldots \cdot L_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ L_{n1} \cdot \ldots \cdot L_{nn} \end{bmatrix} \quad [S] = \begin{bmatrix} S_{11} \cdot \ldots \cdot S_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ S_{n1} \cdot \ldots \cdot S_{nn} \end{bmatrix}.$$

Making use of formula (8) for the transformation of a derivative and (10) for the transformation of an integral, we obtain

$$Q[Z][i] = [Z(p)][g] - [L][i^0] + \frac{1}{p}[S][q^0], \quad . \quad . \quad (31)$$

and our matrix equation (28) becomes

$$[Z(p)][g] = [G] + [L][i^0] - \frac{1}{p}[S][q^0]. \quad . \quad . \quad (32)$$

This is a generalization of equation (11) for the simple series circuit. The only difference is the substitution for the scalar quantities of (11), the matrix quantities found in (32).

If we now multiply equation (32) by the inverse matrix of $[Z(p)]$, $[Z(p)]^{-1}$, we obtain

$$[g(p)] = [Z(p)]^{-1}([G] + [L][i^0] - \frac{1}{p}[S][q^0]). \quad . \quad (33)$$

This is the generalization of (13).

The currents are given explicitly by the matrix equation

$$[i(t)] = \frac{1}{2\pi j} \int_{-j\infty + c}^{+j\infty + c} e^{pt}[g(p)] \, dp. \quad . \quad . \quad (34)$$

This is the generalization of (15).

We have thus by the use of the Laplacian transformation and matrix notation succeeded in obtaining an explicit statement for the currents of a constant linear lumped network having initial charges and currents, and on which are impressed arbitrary electromotive motive forces at $t = 0$.

The simplicity of this method over the usual classical methods may be fully appreciated in the realization that we have our $2n$ arbitrary constants fully determined without further labour. In the numerical solution of the problem the matrix notation assists greatly in systematizing the work. The remarkable conciseness of the matrix notation is evident in that equations (34) and (33) are direct generalizations of the scalar equations (13) and (15).

Two Coupled Circuits.

As an illustration of the method let us briefly consider as a special case the problem of two coupled circuits. Let us suppose the circuits to be represented by the diagram below (fig. 2). There is magnetic coupling between circuits one and two. The electromotive forces are supposed to be impressed at $t=0$ and initial charges and currents will be considered.

The various circuit constants are labelled in the above circuit diagram. Let the initial currents be i_1^0 and i_2^0 and the initial charges be q_1^0 and q_2^0.

Let us further suppose that the impressed e.m.f.'s are sinusoidal, but not necessarily of the same frequency. Therefore

$$e_1 = E_1 \sin w_1 t \quad \text{and} \quad e_2 = E_2 \sin w_2 t.$$

Fig. 2.
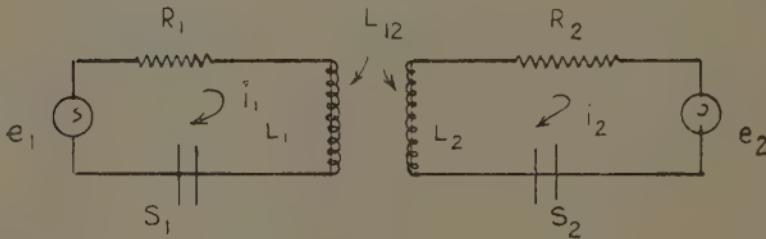
The transforms of these two electromotive forces are easily obtained by integration and are

$$G_1(p) = \frac{E_1 w_1}{P^2 + w_1^2} \quad \text{and} \quad G_2(p) = \frac{E_2 w_2}{p^2 + w_2^2}.$$

In this application we have

$$Z_{11} = pL_1 + R_1 + S_1/p,$$

$$Z_{12} = pL_{12} = Z_{21},$$

$$Z_{22} = pL_2 + R_2 + S_2/p.$$

We now must calculate the matrix $[g]$ given by (33).

$$[g] = [Z]^{-1}([G] + [L][i^0] - \frac{1}{p}[S][q^\circ]). \quad . \quad . \quad (33)$$

By matrix multiplication we readily find $[g]$ to be

$$[g] = \frac{1}{|Z|} \left\| \begin{array}{c} (Z_{22}G_1 - Z_{12}G_2) \\ (-Z_{21}G_1 + Z_{11}G_2) \end{array} \right\|$$
$$+ \frac{1}{|Z|} \left\| \begin{array}{c} ([L_1Z_{22} - L_{12}Z_{12}]i_1^0 + [Z_{22}L_{12} - Z_{12}L_2]i_2^0) \\ ([-Z_{21}L_1 + Z_{11}L_{12}]i_1^0 + [-Z_{21}L_{12} + Z_{11}L_2]i_2^0) \end{array} \right\|$$
$$- \frac{1}{p|Z|} \left\| \begin{array}{c} (Z_{22}S_1q_1^0 - Z_{12}S_2q_2^0) \\ (-Z_{21}S_1q_1^0 + Z_{11}S_2q_2^0) \end{array} \right\|,$$

where

$$|Z(p)| = \left| \begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array} \right| = Z_{11}Z_{22} - Z_{21}Z_{12},$$

or

$$|Z(p)| = \frac{1}{p^2}([L_1L_2 - L_{12}^2]p^4 + p^3[L_1R_2 + L_2R_1]$$
$$+ p^2[L_1S_2 + L_2S_1 + R_1R_2] + p[R_1S_2 + R_2S_1] + S_1S_2),$$

and hence the zeros of $|Z(p)|$ depend upon the roots of a quartic equation. If numerical values are available for the constants this equation may be solved by the usual method. The residues of $g_1(p)$ and $g_2(p)$ may then be readily calculated and the two mesh currents i_1 and i_2 obtained. The procedure here may appear complicated. However, it must be realized that once the solution is obtained no bothersome arbitrary constants need be considered. The effect of each initial condition and e.m.f. is clearly seen in equation (33). The solution of a less general consideration of the coupled circuit here considered is given in ' Electric Oscillations and Electric Waves ' by G. W. Pierce (McGraw-Hill), pp. 94–137. The advantages of the present method will be evident if consultation of this treatment be made.

L. *Transition from Hexagonal to Cubic Electrodeposited Chromium.* By W. A. Wood, *M.Sc., Physics Department, National Physical Laboratory, Teddington, Middlesex* *.

[Plates I. & II.]

THE atomic structure of electrodeposited chromium may be either of the body-centred cubic or the hexagonal type [1]. The former is the more stable ;

the hexagonal structure passes irreversibly into the cubic on heating.   The present work was undertaken with the object of determining exactly the temperature of transition, and of comparing, by the method of X-ray diffraction, the physical condition of the cubic product thus formed with that of a straightforward cubic deposit. A practical aspect of the work arises from the fact that the hexagonal form, being much softer than the cubic, represents a deleterious factor in chromium plating, the technical value of which lies mainly in its abnormally high hardness.   It would be useful, therefore, to remove the hexagonal material if the cubic formed from it was of the hard type.

Small amounts of the hexagonal formed in ordinary cubic deposits are difficult to detect because the X-ray method, which affords the only direct means of analysis, is not sensitive in this particular case, owing to the submicroscopic size of the crystallites forming the deposit. This small size gives rise to a characteristically diffuse type of spectrum in which it is often impossible to detect the contribution of less than 10 per cent. impurity.   This lack of sensitivity, together with the difficulty of controlling plating conditions, is probably the reason for the con-flicting reports made by earlier workers [2] on the presence or absence of hexagonal chromium in ordinary plating, which normally gives a preponderance of the cubic form. On theoretical grounds, however, it is to be expected that traces at least of the hexagonal form are inevitable.   For, as suggested in previous work [3], and confirmed recently by Wright, Hurst, and Riley [4], the criterion for its formation is the valency of the chromium in the electro-lyte, the *chromic* ion giving the cubic and the *chromous* the hexagonal variety.   Since the large evolution of hydrogen characteristic of the standard chromic acid bath is likely to lead to some reduction, the deposition of some proportion of hexagonal material is therefore highly probable.

Procedure.

Specimens.

As indicated above, it is necessary to add a reducing agent to the chromic acid bath in order to produce a definite preponderance of the *chromous* ions and therefore of the hexagonal structure.   In the present experiments

this was done by addition of sugar, and the bath finally used as standard had the composition

	per litre:
CrO_3	600 gram.
H_2SO_4	0·075 N.
Sugar	10 gram.

The chromium was deposited on 12-inch lengths of copper strip, $\frac{1}{2}$ inch by $\frac{1}{8}$ inch section, after the surface had been cleaned and lightly etched. The current density employed was of the order of 0·2 amp. per sq. cm., and the temperature of the bath 18°–21° C. These conditions gave a uniform lustrous coating which was found to consist almost entirely of the hexagonal structure.

Heat Treatment.

The aim of the investigation was to examine specimens which had been held at different temperatures for a known period of time. It was decided that this would be achieved most simply by taking one specimen and establishing along the bar a measured temperature gradient for a specific time. An examination from point to point along the bar for cubic and hexagonal chromium then showed the position at which transition from one form to the other had taken place. The temperature to which this place had been heated was determined from the temperature gradient curve. The process was next applied to further specimens similarly heated but for a different length of time. In this way the effect of the time of heating on the transition temperature could be found in a sensitive way.

The temperature gradient was set up by clamping the end of the bar to be heated between two sheets of mica, 2 inches by $\frac{1}{2}$ inch, wound with nichrome wire and suitably lagged with asbestos. The cold end of the bar was kept at a constant low temperature by water-cooling. Actually each bar consisted of two similar plated specimens fastened face to face by screws, so that thermocouples could be conveniently inserted between the surfaces at intervals along the length of the composite bar, and the temperature of those points ascertained. Under these conditions a steady state could be reached after about three minutes. Specimens were subjected to a temperature gradient for 5, 15, or 60 minutes ; the gradients applied to two typical specimens, heated

respectively for 15 and 60 minutes, are shown in fig. 1.
It will be seen that the temperature along these specimens
decreased from approximately 160° to 20° C., a range
which was found by preliminary experiments completely
to include the region of the required transition.

X-ray Examination : –(*a*) In order to determine the
transition point each plated specimen was set up in a

Fig. 1.
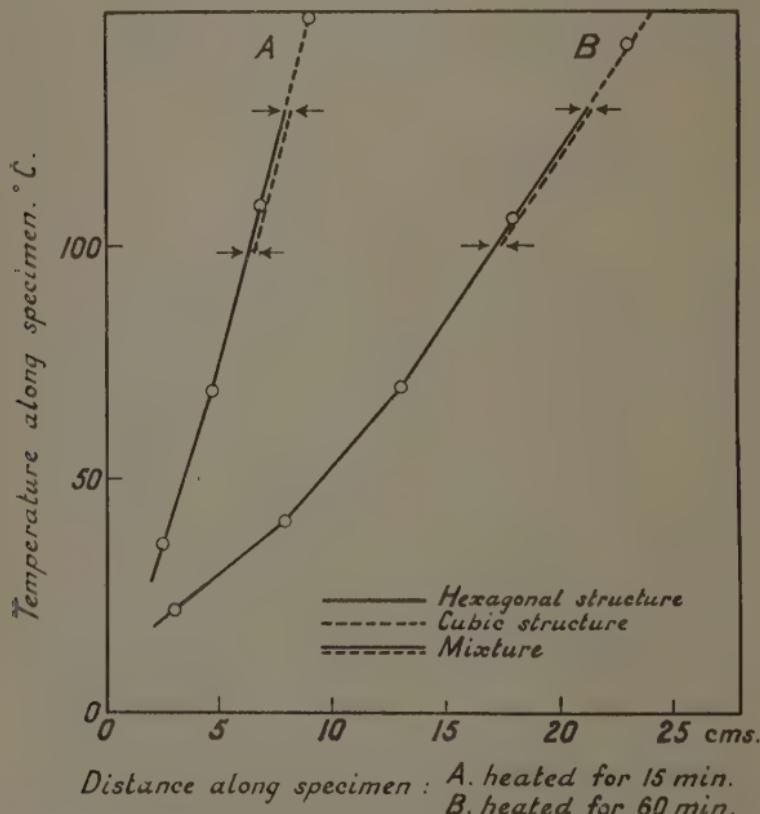
Distance along specimen : A. *heated for 15 min.*
B. *heated for 60 min.*

Showing transition from hexagonal to cubic chromium plate on heating.

circular X-ray camera and diffraction photographs
obtained from the surface at intervals of 1 cm. along the
length. Beginning at the cold end, the photographs
gave the characteristic spectrum of the hexagonal
material, then a composite spectrum containing the lines
of the hexagonal together with those of the cubic struc-
ture, and, finally, at the heated end, the cubic alone.
X-ray photographs showing these changes are reproduced
in fig. 2 (Pl. I.) ; the more highly resolved end of the

spectrum is illustrated, since the lines then show also the broadening effect referred to below.

(*b*) The hardness of the chromium formed by heating in comparison with that of a normal deposit was judged from the line breadth of the spectral lines. Sufficient evidence has accumulated in the past few years, from work on the hardness of cold-rolled metals, metals hardened by nitriding, and electro-deposited metals [5] to indicate that there is a relationship between the hardening and the line-broadening of the X-ray spectral lines ; the latter effect is a measure of the dislocation or distortion of the metallic grain into constituent crystallites, which appear to be associated with the hardening.

For this purpose back-reflexion photographs were obtained from the positions giving a mixture of the two structures, so that the diffraction rings characteristic of both hexagonal and cubic material were obtained on the same photograph, and variables due to possible differences in processing the negatives were thus obviated. Other photographs were obtained from each structure alone and from cubic deposits obtained directly by ordinary deposition.

The interpretation of the line-breadths in terms of size of crystallites was made by Scherrer's formula, which is sufficiently accurate for the purpose. If δB is the increase in breadth due to reduction of crystallites to the mean linear size t, we have

$$\delta B = \frac{0 \cdot 9 \, \lambda}{t \cos \theta}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

where λ is the wave-length used and θ the diffraction angle. The diffraction ring from the cubic chromium used for measurement was the (211) ring for which $\theta_2 = 76 \cdot 4°$; for the hexagonal the (112) was employed for which $\theta_1 = 80 \cdot 7°$.

If the crystallite size of the hexagonal material was the same as that of the cubic formed from it, then, owing to the difference in diffraction angles, the breadth of the hexagonal ring should be some 1·5 times greater than that of the cubic, according to the formula. Actually it was found that the reverse was the case ; the hexagonal line was distinctly narrower than the cubic, so that there could be no doubt that the hexagonal crystallites were therefore much larger than the cubic crystallites to which

they gave rise. This comparison is shown by the repro-
duction of the relevant diffraction rings in fig. 3 (Pl. II.).

RESULTS.

Transition Temperature.—(1) In the case of specimens
heated for 1 hour and then examined, the first definite
appearance of the cubic material, arising out of the
initial hexagonal, was made at 100° C. Between 100°
and 130° C. the proportion of cubic structure rapidly
increased, and at the latter temperature the transition
was entirely complete. There is, therefore, no sudden
change-over from one to the other, but an extended
transformation over a range of some 30° C.

(ii.) In the specimens heated for 15 minutes the change
began at 98° C. and was concluded at 129° C. Com-
parison with the preceding specimen therefore indicated
that four-fold increase in the time of heating produced
no narrowing of the range of temperature over which
the transition to the cubic form took place. It is possible
that a very long heating might have accelerated this
change, but no advantage would be gained, since a shorter
time of heating at the slightly higher temperature would
always be preferable. The above results are depicted
in fig. 1.

(iii.) In the specimens heated nominally for 5 minutes
the time required to establish a steady state was a high
proportion of the whole, so that the exact time of heating
was indeterminable. The X-ray photographs showed,
however, that the change to the cubic structure took
place over the same temperature range as the above
specimens.

The above results indicate, then, that a heat treatment
of about 15 minutes at 130° C. is quite sufficient to
remove any hexagonal structure formed in a deposit
and that more extended heating is unnecessary.

Change in Physical Condition.—From the Scherrer
formula, (1) we have for the ratio between the crystallite
size of the cubic and hexagonal material,

$$\frac{t_1}{t_2} = \frac{\delta B_2}{\delta B_1} \cdot \frac{\cos \theta_2}{\cos \theta_1} = \frac{\delta B_2}{\delta B_1} \times 1\cdot5,$$

where the suffix (1) refers to the hexagonal and the
suffix (2) to the cubic structure. The radial breadths
of the diffraction rings were measured with the aid of

a Moll microphotometer; the intrinsic breadth which the rings would have possessed had they proceeded from normal materials with a size and perfection of grain giving sharply resolved lines was known from calibration experiments with such materials; the difference between this latter breadth and that of the diffuse lines under investigation gave the relevant values of δB for insertion in the above equation. Averaging the observations over a number of specimens it was found that

$$\frac{\delta B_2}{\delta B_1} = 1 \cdot 8,$$

whence

$$\frac{t_1}{t_2} = 2 \cdot 7.$$

Hence when the hexagonal type of chromium plate is transformed by the low temperature heat treatment described the cubic structure thus formed has approximately one-third the crystallite size of the initial material.

The absolute crystallite size of these structures was found by substitution in formula (1). The values obtained were

$$t_1 = 4 \cdot 0 \times 10^{-6} \text{ cm.}$$
$$t_2 = 1 \cdot 5 \times 10^{-6} \text{ cm.}$$

The absolute value of the crystallite size of cubic chromium plating X-rayed under exactly the same conditions as the above specimens was similarly measured, and, by application of the Scherrer formula gave values ranging from 1×10^{-6} to $1 \cdot 7 \times 10^{-6}$ cm., according to the current density employed; the smaller values characterizing the bright type of plating and the larger the dull type. The cubic product of transformation formed by heating the hexagonal material has therefore the same order of crystallite size as cubic chromium deposited directly. The similarity of the line-broadening in the two cases is illustrated in fig. 3 (Pl. II.). It may be possible by varying conditions of deposition to produce chromium deposits with grain size of a different order from the above values; but, using the standard chromic acid baths, it is found that the sizes of the crystallites cover a remarkably small range, so that the above values may be regarded as typical of those met in practice. In no instance has a normal chromium plated specimen been found giving a diffraction ring with so small a breadth as that given by hexagonal chromium.

In view of the general relationship between the diffusion of the diffraction rings and the hardness of the metal, to which reference has been made, it follows that the cubic product of the transformation is definitely harder than the original hexagonal structure, and that the increase in hardness brings it into the class of the directly deposited cubic material which preponderates in ordinary chromium plate. The softening effect of the hexagonal constituent can therefore be removed by the short heat-treatment without appreciably affecting the state of the remainder *.

At present there is not available any certain way of determining the hardness of the thin metallic coatings used in chromium plating. It may be possible to obtain hardness measurements, however, on thick deposits which will permit of quantitative expression of the above changes, indicated by the X-ray method, and it is hoped soon to secure data of this kind.

SUMMARY.

A determination is made of the exact temperature range over which electroplated chromium with hexagonal atomic structure undergoes transformation to the cubic form. It is found that the X-ray diffraction rings exhibit a distinct increase in breadth when comparison between the spectra of the cubic product and the initial hexagonal material is made. The decrease in the crystallite size causing this effect is measured, as well as the characteristic crystallite size of the hexagonal chromium and directly deposited cubic chromium. It is shown that the cubic chromium produced by the transformation will be harder than the hexagonal type and resemble the normal cubic deposit, and that therefore the softening effect of any hexagonal constituent in a deposit may be effectively removed.

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(2) C. S. Smith, Met. Ind. (London), xxviii. p. 456 (1926) ; F. Sillers, Trans. Am. Electrochem. Soc. lii. p. 301 (1927).
(3) W. A. Wood, Phil. Mag. xii. p. 853 (1931).
(4) L. Wright, H. Hirst, and J. Riley, P. 1253, " Discussion Metallic Coatings," Trans. Farad. Soc., September 1935.
(5) W. A. Wood, Trans. Far. Soc. xxxi. p. 1248 (1935).

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* It is important to keep a low heating temperature, otherwise the cubic chromium rapidly softens on its own account.

LI. *Notices respecting New Books.*

The Organic Chemistry of Nitrogen. By N. V. SIDGWICK, F.R.S.
New Edition. Revised and re-written by T. W. J. TAYLOR
and WILSON BAKER. [Pp. 573.] (Oxford : University
Press, 1937. Price 25s. net.)

THERE is probably fairly general agreement among British
chemists that one of the most outstanding scientific
events of 1928 was the publication of Sidgwick's 'Electronic
Theory of Valency.' Many will hold the opinion that the
appearance of the new edition of ' The Organic Chemistry of
Nitrogen ' is of almost equal importance. Although the book
will not, on account of its somewhat more specialized character,
be of quite such general interest to all chemists, the theoretical
treatment from the physico-chemical and the electronic stand-
points will be appreciated by many who have never carried
out research in organic chemistry. In fact, as regards the
electronic treatment of the subject, parts of this work might be
regarded as a supplement to Professor Sidgwick's more general
treatise.

One of the most important features of the work is the
relatively simple structure of most of the compounds with which
it deals—amines, nitro-, cyano-, and thiocyano-derivatives,
etc., to mention only a few. This is again noticed in the
treatment of the heterocyclic systems : pyrrole, indole,
pyridine, quinoline, and isoquinoline are discussed, but no
compounds with more than one cyclic nitrogen atom are
considered.

The authors believe that the space saved by omitting any
treatment of the purines and the alkaloids has been better
utilized by a more fundamental consideration of the simpler
types of compounds. This decision will be approved by almost
every reader of the work. It may also be regarded as having
been dictated by the manner in which organic chemical
research has recently developed. Many of the important
advances in theoretical organic chemistry of the last fifteen
years are due to the application of the electronic theory
of chemical reaction to a consideration of the structure of,
necessarily, simple molecules. Again, the study of structure
by the newer methods of physical chemistry, such as the deter-
mination of dipole moment, X-ray analysis, and the diffraction
of electrons has yielded some of the most interesting results
in the case of relatively simple molecules.

The method of treatment chosen by the authors is well
exemplified by their discussion of recent work on the con-
stitution of the isomeric oximes, of the aliphatic diazo-
compounds, and of the triazo-derivatives. The long drawn-
out discussions regarding the relative merits of the cyclic and
open chain formulæ for the last two classes of compounds

would now appear to have been closed, and the structure of compounds in question must be regarded as a resonance hybrid between the possible open chain structures. It should be mentioned, in passing, that Professor Sidgwick has included an introduction, written by himself, in which he deals with the importance of the theory of resonance in organic chemistry.

More striking instances than those just quoted could hardly have been found to illustrate the value of the new methods of attack now at the disposal of the organic chemist—all of which have been perfected, if not entirely devised, during the period which has elapsed since the first edition of this work was published.

These methods have been equally valuable in the determination of the configuration of the isomeric oximes, but it is interesting to recall, as the authors point out, that the revolution in our ideas on this subject arose from the work of Meisenheimer, who employed the standard methods of the organic chemist.

Turning to other aspects of the work the short reference to the researches of Krebs on the mechanism of urea formation in the liver is to be welcomed, since the recent advances in organic chemistry on the biological side are not less noteworthy than those, possibly better known to many workers, which have been made on the borderland between physics and chemistry. In dealing, however, with the amine oxides, the betaines and the tetraalkyl ammonium hydroxides, the work of Ackermann, which has established the occurrence of many such compounds in marine animals, might have been mentioned.

Other features of the book which the reviewer has found of particular interest are the discussion of the possible structure of the picrates of the aromatic hydrocarbons, and of the polymers arising from the pyrrole. The views expressed on the structure of these complex heterocyclic compounds are based on a modification of the Diels-Alder reaction and illustrate the increasing usefulness of this reaction not only in synthetic work but also in interpreting the mode of formation of complex molecules.

The book is beautifully produced, as would be expected in the case of a publication from the Oxford University Press, and is written in a very clear and pleasing style.

It will be found of great assistance to the advanced student who wishes to supplement his lecture notes, to the lecturer temporarily non-plussed by a question from an enthusiastic student, to the research worker looking for analogies to some reaction he is studying, and not least to the senior chemist planning a new line of work.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 2.
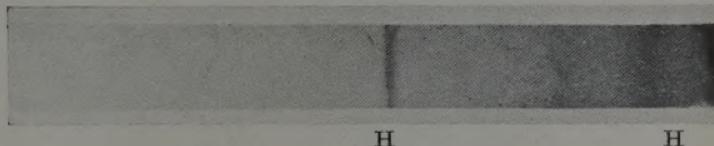
95° C.   Hexagonal only (H).
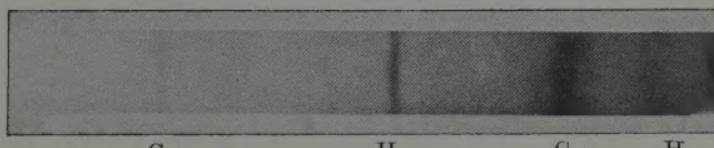
$\left.\begin{array}{l}117°\ C. \\ 125°\ C.\end{array}\right\}$ Hexagonal + cubic.
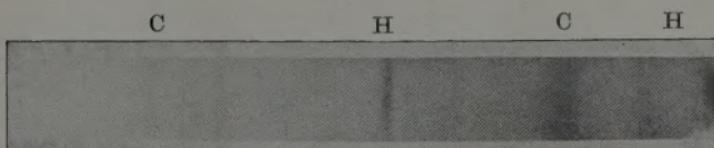

133° C.   Cubic only (C).


Change from hexagonal to cubic chromium.

Fig. 3.
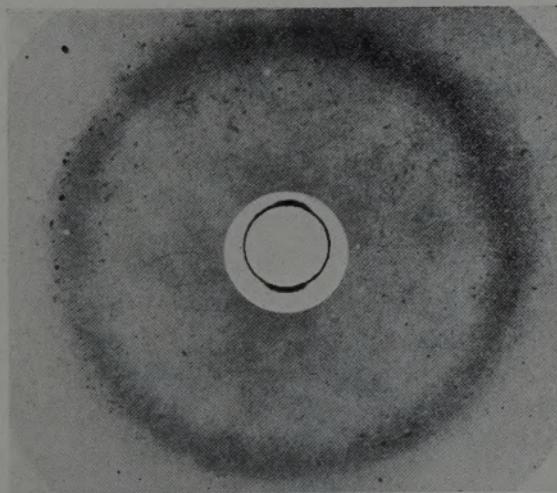
Cubic chromium
(from heating
hexagonal).
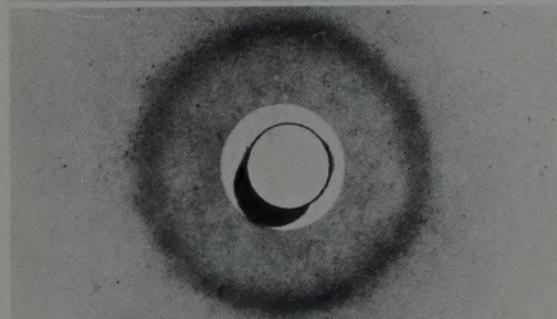
Hexagonal
(direct deposit).
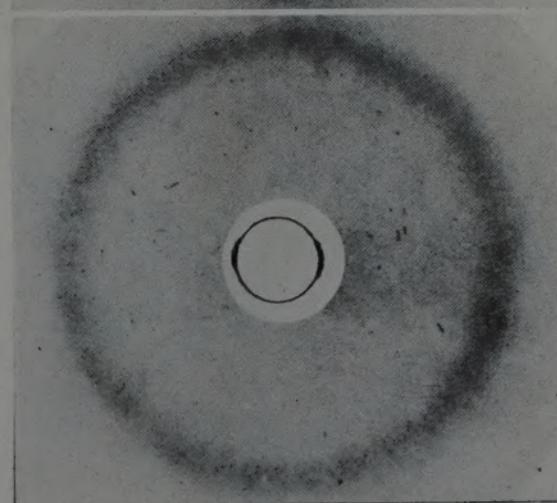
Cubic
(a direct deposit).

Showing increase in width of halo on passing from
hexagonal to cubic chromium.